GOVERNMENT POLYTECHNIC, GAJAPATI

DEPARTMENT OF MECHANICAL ENGG

STUDY MATERIAL

STRENGTH OF MATERIAL (TH-2)

3RD SEMESTER

MECHANICAL ENGG.

BY

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CONTENTS:

Course outcomes

At the end of the course students will be able to:

Course Outcomes(CO)

C202.1: Able to explain and analyse basics of different types of stress and strain .

C202.2:Able to draw and analyse shear force diagram and bending moment diagram of various types of beams under different types of loading conditions.

C202.3:Able to explain theory of simple bending and torsion equation.

C202.4:Able to analyse two dimensional stress system,thin cylinder and spherical shell under internal pressure.

MODULE-1

Load →

Load may be defined as the combined effect of external forces acting on body

Civil ÷ Forces which are applied to a structure.

Mechanical ÷ The external mechanical resistance against which a machine, such as motor or engine, acts

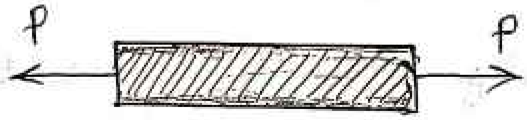
Electrical ÷ A device connected to the output of a circuit

Load is an external agent which causes deformation of the body.

Loads may be classified as
i) Dead Load
ii) Fluctuating Load
iii) Inertia load
iv) Centrifugal load

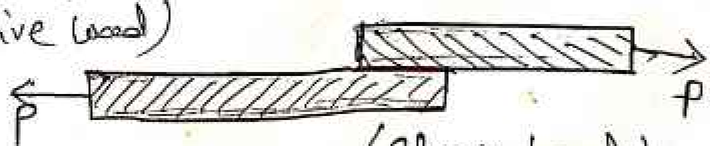
Loads may also be catagorised as
i) Tensile Load
ii) Compressive Load
iii) Shear Load
iv) Torsional load or twisting load
v) Bending load.

Loads may be
i) Point load
ii) Uniformly distributed load
iii) Uniformly varying load



(Tensile load)

(Compressive load)



(Shear load)

Stress →

Whenever a body is subjected to Loads, an internal resisting force will develope to oppose deforemation. This resisting force per unit c/s area is known as stress.

$$\text{Stress} = \frac{\text{Resisting force}}{\text{c.s. area}}$$, it is represented as 'σ' (Sigma)

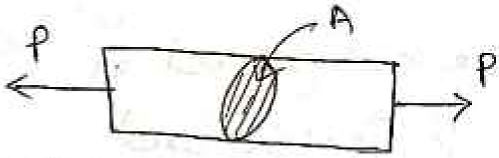
Unit is N/mm^2

Stress may be classified into three types

i) Tensile stress
ii) Compressive stress
iii) Shear stress

Tensile stress →

* When a body is subjected to two equal and opposite pulls, it elongates. The internal resisting force due to this pulls per unit cross-sectional area is known as Tensile stress.

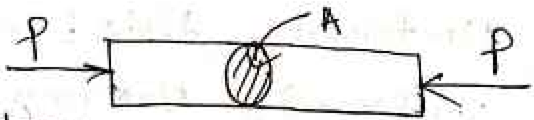


$$\sigma_T = \frac{P}{A}$$

P = Tensile load
A = C.S. area

Compressive stress →

* When a body is subjected to two equal and opposite push, it contracts. The internal resisting force due to these push per unit cross-sectional area is known as compressive stress.

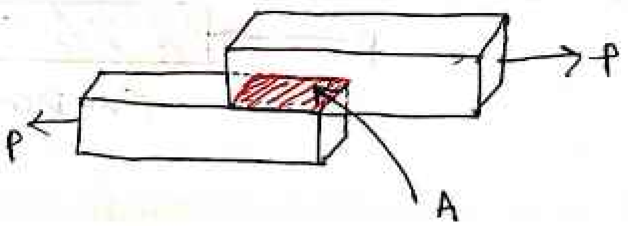


$$\sigma_c = \frac{P}{A}$$

Shear stress →

* When the external force on body tries to shear the body, this shear force per unit C.S. area is known as shear stress.

$$\sigma_s = \frac{P}{A}$$

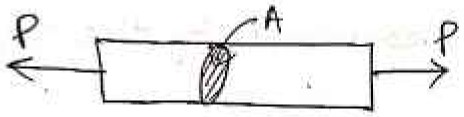


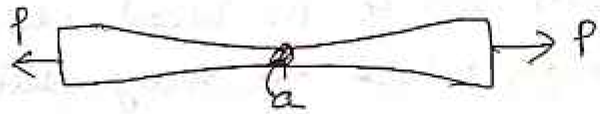
Engineering stress & True stress →

When a body undergoes external loads, it deforms and an internal resisting force is developed.

This force per original cross-sectional area is known as Engineering stress.

During deformation, if the stress is to be calculated at a particular instant, That resisting force per cross sectional area at that instant is known as True Stress.





$$\sigma_{Engg.} = \frac{P}{A} \qquad\qquad \sigma_{True} = \frac{P}{a}$$

Strain

When a material is subjected to stress, it is said to be strained i.e it undergoes deformation. This deformation per original length is known as Strain

$$\text{Strain}(\epsilon) = \frac{\text{change in length}}{\text{original length}} = \frac{\Delta l}{L}$$

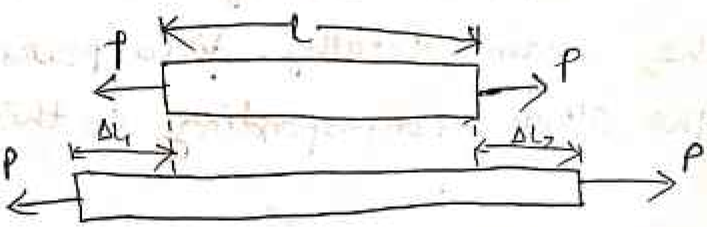
Strain may be classified as
i) Tensile strain
ii) Compressive strain
iii) Shear strain
iv) Volumetric strain

Tensile strain

* When a body is subjected to tensile stress, it elongate i.e. This increment of length is called deformation of that material. This increment length per original length is known as Tensile strain.

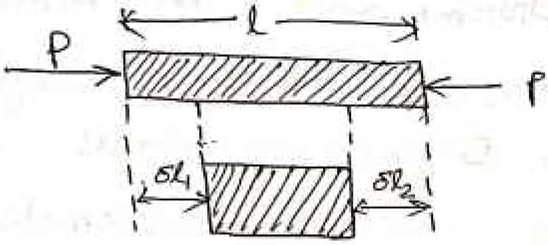
$\Delta l = \Delta l_1 + \Delta l_2$

$e_L = \frac{\Delta l}{l}$



Compressive strain

* When the body is subjected to compressive stress, it contracts. This decrement of length per original length is known as compressive strain.

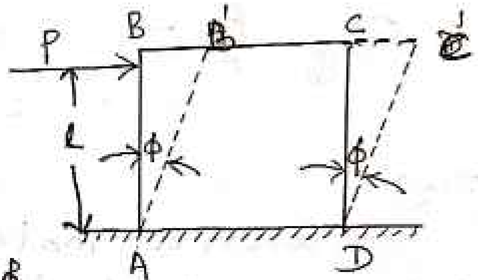


$$\delta l = \delta l_1 + \delta l_2$$

$$\varepsilon_c = \frac{\delta l}{l}$$

* Shear strain

When a body is subjected to shear load, shear strain will be produced which is measured by the angle through which the body distorts.



$$\varepsilon_s = \tan\phi$$

$$= \phi \quad \text{(Since } \phi \text{ is very small)}$$

Note: Shear strain = $\frac{\text{Deformation}}{\text{original length}}$

$$\varepsilon_s = \frac{CC' \text{ or } BB'}{L} = \tan\phi$$

* Volumetric strain

It is defined as the change in volume to the original volume of the body.

$$\varepsilon_v = \frac{\Delta V}{V}$$

Elastic & Plastic body →

When the body will regain its ~~sh~~ original shape and size after the removal of load (external) then that body is known as elastic body.

The body is said to be plastic when the strains exist even after the removal of external force.

There is always a limiting value of load upto which the strain totally dissappears on the removal of load. The stress corresponding to this load is called elastic limit.

Hooke's Law →

According to Hooke's Law, within elastic limit, stress varies directly with strain.

i.e. Stress $\propto$ Strain (within elastic limit)

$$\Rightarrow \frac{\text{Stress}}{\text{Strain}} = \text{Constant}.$$

This constant is known as Modulus of elasticity.

✱ Young's Modulus →

It is defined as the ratio of tensile stress to tensile strain or compressive stress to compressive strain. It is denoted as 'E'. It is equal to the modulus of elasticity.

$$E = \frac{\sigma}{\varepsilon}$$

✱ Modulus of Rigidity →

It is defined as the ratio of shear stress to the shear strain. It is denoted as G or C. It is also called shear modulus of elasticity.

$$G = \frac{\tau}{\varepsilon_s}$$

✱ Bulk modulus or volume modulus of elasticity →

It may be defined as ratio of normal stress to volumetric strain. It is denoted as K.

$$K = \frac{\sigma_n}{\varepsilon_v}$$

Poisson's Ratio →

If a body is stressed within its elastic limit, the lateral strain bears a constant ratio to linear strain.

$$\frac{\text{Lateral Strain}}{\text{Linear Strain}} = \text{Constant} \quad \left(\frac{1}{m} \text{ or } \mu\right)$$

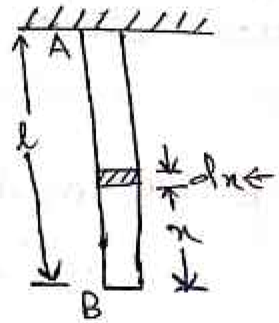
Principle of superposition →

Sometimes a body is subjected to a number of forces acting on its outer edges as well as at some other sections, along the length of the body. In such a case, the forces are split up & their effects are considered on individual sections. The resulting deformation of the body, is equal to the algebric sum of the deformations of the individual sections. Such principle of finding out the resultant deformation, is called principle of superposition.

$$\Delta l = \frac{1}{AE}\left[P_1 l_1 + P_2 l_2 + \cdots\right]$$

Deformation of a body due to self weight →

Consider a bar AB hanging freely under its own weight as shown in fig.



let l = length of the bar

A = Cross-sectional area of the bar

E = Young's modulus for the bar

w = specific weight of bar $= \frac{wt.}{Volm}$

Now consider a small section 'dx' of the bar at a distance 'x' from B. We know wt. of the bar for a length of 'x'.

$$P = wAx$$

elongation of the small section of bar due to wt. of the bar for the length of x $= \frac{Pl}{AE} = \frac{(wAx)dx}{AE}$

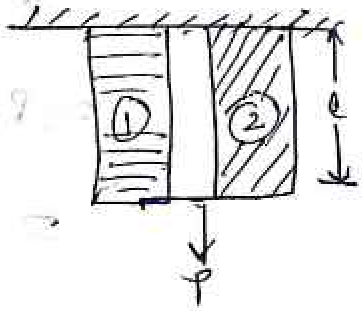
$$= \frac{wx\,dx}{E}$$

Total elongation $\delta l = \int_0^l \frac{wx\,dx}{E} = \frac{wl^2}{2E}$

Stress in composite bars →

A bar made up of two or more different materials, joined together is called a composite bar. The bars are joined in such a manner, that the system expands or contracts as one unit, equally when subjected to tension or compression.

1) Extension or contraction of the bar being equal, the strain i.e deformation per unit length is also equal.
2) The total external load, on the bar, is equal to the sum of the loads carried by the different material.



$$P = P_1 + P_2$$

Consider a composite bar made up of two different materials as shown in fig.

P = Total load on the bar

l = length of the bar

A_1 = area of bar 1.

E_1 = modulus of elasticity of bar 1

P_1 = Load shared by bar 1.

A_2, E_2, P_2 = Corresponding values of bar - 2

Total load on the bar

$$P = P_1 + P_2$$

Stress in bar 1. $\sigma_1 = \frac{P_1}{A_1}$

Strain in bar 1. $\epsilon_1 = \frac{\sigma_1}{E_1} = \frac{P_1}{A_1 E_1}$

Elongation $\Delta l = \epsilon_1 . l = \frac{P_1}{A_1 E_1} l$

Similarly elongation in bar 2, $= \frac{P_2 l}{A_2 E_2}$

Since both of the elongations are equal.

$$\frac{P_1 L}{A_1 E_1} = \frac{P_2 L}{A_2 E_2}$$

$$\frac{P_1}{A_1 E_1} = \frac{P_2}{A_2 E_2}$$

$$P_2 = \frac{P_1 A_2 E_2}{A_1 E_1}$$

$$P = P_1 + P_2$$

$$= P_1 + \frac{P_1 A_2 E_2}{A_1 E_1}$$

$$= P_1 \left[1 + \frac{A_2 E_2}{A_1 E_1} \right]$$

$$= P_1 \left[\frac{A_1 E_1 + A_2 E_2}{A_1 E_1} \right]$$

$$P_1 = \frac{P \times A_1 E_1}{[A_1 E_1 + A_2 E_2]}$$

Similarly $\quad P_2 = P \times \dfrac{A_2 E_2}{A_1 E_1 + A_2 E_2}$

Again, $\quad \dfrac{P_1}{A_1 E_1} = \dfrac{P_2}{A_2 E_2}$

$$\frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2}$$

$$\sigma_1 = \frac{\sigma_2}{E_2} \times E_1$$

$$\sigma_1 = \frac{E_1}{E_2} \sigma_2$$

Similarly $\quad \sigma_2 = \dfrac{E_2}{E_1} \sigma_1$

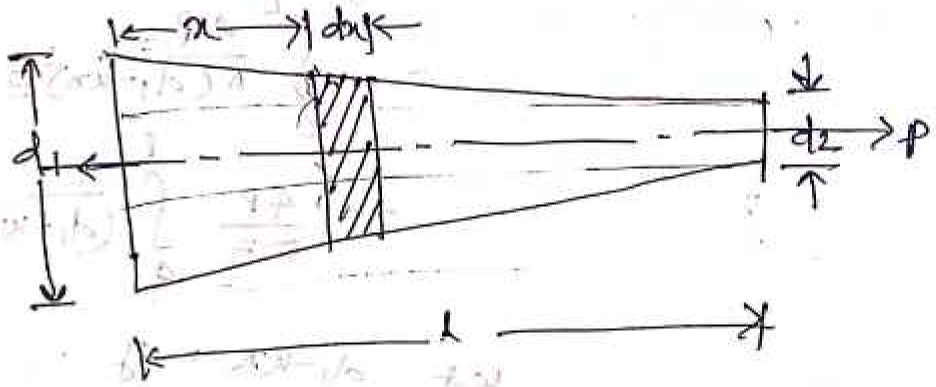
$\dfrac{E_1}{E_2}$ = Modular ratio & denoted by 'm'.

$$P = P_1 + P_2$$

$$\boxed{P = \sigma_1 A_1 + \sigma_2 A_2}$$

Stresses in bars of uniformly Tapering section →

Consider a circular bar of uniformly tapering section as shown in fig.



P = Pull on the bar

L = Length of the bar

d_1 = Diameter of the bigger end of the bar

d_2 = Diameter of the smaller end of the bar

Consider a small element of length dx, of the bar at a distance 'x' from the bigger end.

Diameter of the bar at a distance 'x' from the left end.

$$d_x = d_1 - (d_1 - d_2)\frac{x}{l}$$

$$= d_1 - Kx \qquad \left(\text{where } K = \frac{d_1 - d_2}{l}\right)$$

$d_1 - (d_1 - d_2)$

$l \rightarrow d_1 - d_2$

$1 \rightarrow \frac{d_1 - d_2}{l}$

$x \rightarrow \frac{d_1 - d_2}{l} \times x$

$$A_x = \frac{\pi}{4}(d_1 - Kx)^2$$

$$\sigma_x = \frac{P}{A_x} = \frac{P}{\frac{\pi}{4}(d_1 - Kx)^2} = \frac{4P}{\pi(d_1 - Kx)^2}$$

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{\frac{4P}{\pi(d_1 - Kx)^2}}{E} = \frac{4P}{\pi(d_1 - Kx)^2 E}$$

Elongation of the elementary length (10)

$= \varepsilon_x \, dx$ $\quad \left(\varepsilon_x = \frac{dx'}{dx}, \quad dx' = \varepsilon_x dx \right)$

Total elongation of the bar may be found out by integreating the above

$$\delta l = \int_0^l \varepsilon_x \, dx$$

$$= \int_0^l \frac{4P}{\pi (d_1 - kx)^2 E} \, dx$$

$$= \frac{4P}{\pi E} \int_0^l \frac{1}{(d_1 - kx)^2} \, dx$$

let $d_1 - kx = y$

$-k\,dx = dy$

$dx = -\frac{1}{k} dy$

when $x \to 0$, $y \to d_1$

$x \to l$, $y \to d_1 - kl$

$$= \frac{4P}{\pi E} \int_{d_1}^{d_1 - kl} \frac{1}{y^2} \times -\frac{1}{k} \, dy$$

$$= -\frac{4P}{\pi E k} \int_{d_1}^{d_1 - kl} y^{-2} \, dy$$

$x^n \, dx$

$\frac{x^{n+1}}{n+1}$

$$= \frac{-4P}{\pi E k} \left[\frac{y^{-2+1}}{-2+1} \right]_{d_1}^{d_1 - kl}$$

$$= \frac{4P}{\pi E k} \left[\frac{1}{d_1 - kl} - \frac{1}{d_1} \right]$$

Put $k = \frac{d_1 - d_2}{l}$

$$= \frac{4P}{\pi E \frac{(d_1-d_2)}{L}} \left[\frac{\cancel{d_1} - \cancel{d_1} + kl}{d_1(d_1 - kL)} \right]$$

$$= \frac{4PL}{\pi E(d_1-d_2)} \left[\frac{\frac{d_1-d_2}{L} \times l}{d_1 \left[d_1 - \left(\frac{d_1-d_2}{L} \right) \times l \right]} \right]$$

$$= \frac{4PL}{\pi E(d_1-\cancel{d_2})} \left[\frac{(d_1-\cancel{d_2})}{(d_1 \times d_2)} \right]$$

$$\boxed{\delta l = \frac{4PL}{\pi E(d_1 d_2)}}$$

If the bar of uniform diameter 'd' throughout

$$\delta l = \frac{4PL}{\pi E d^2} \qquad (d_1 = d_2 = d)$$

$$= \frac{P}{\left(\frac{\pi}{4} d^2\right)} \times \frac{l}{E}$$

$$\boxed{\delta l = \frac{Pl}{AE}}$$

Thermal Stresses & Strains →

Whenever there is some increase or decrease in the temperature of a body, it causes the body to expand or contract. If the body is allowed to expand or contract freely then no stresses will induced in the body with the rise or fall of temperature. But if the deformation of the body is prevented, some stresses are induced in the body. Such stresses are called thermal stresses or temperature stresses. The corresponding strains are called thermal strains or temperature strains.

Thermal Stresses in simple bar →

The Thermal Stress can be found out

i) Calculating the amount of deformation due to change of temperature with the assumption that bar is free to expand or contract.

ii) Calculating the load required to bring the deformed bar to the original length.

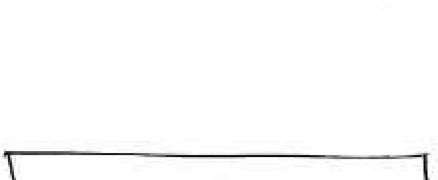
iii) Then stress & strain can be calculated from this load.

OR

1) First find out the deformation due to change in temp.
2) Find out thermal strain due to the deformation.
3) Now, the stress can be found out from strain (using 'E')

Let l = original length of body

t = Increase in temp..



α = Co-efficient of linear expansion. (unit = $/°C$)

$\Delta l = l \alpha t$

If ends of bar is fixed to rigid supports, its expansion is prevented, then compressive strain induced in the bar

$$\varepsilon = \frac{\Delta l}{l} = \frac{l\alpha t}{l} = \alpha t$$

Stress $\sigma = \varepsilon \cdot E$

$$\boxed{\sigma = \alpha t E}$$

Case-1 If the support yield by an amount 'δ'. Then the actual expansion,

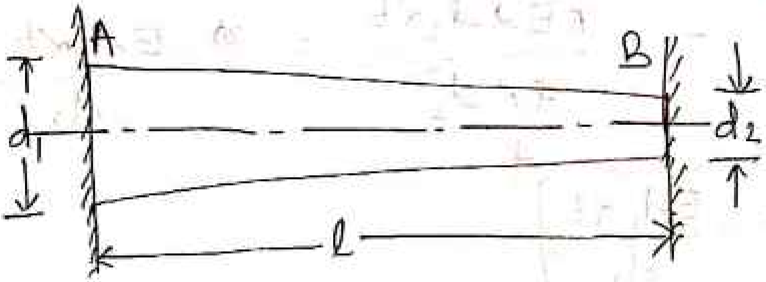
$$\delta l = l\alpha t - \delta$$

$$\delta l = \Delta l - \delta$$
$$= l\alpha t - \delta$$

$$\varepsilon = \frac{\delta l}{l} = \frac{l\alpha t - \delta}{l} = \alpha t - \frac{\delta}{l}$$

$$\boxed{\sigma = \varepsilon E = \left(\alpha t - \frac{\delta}{l}\right) E}$$

Thermal stresses in Bars of uniformly Tapering section →



Consider a circular bar of uniformly tapering section fixed at its ends A & B & subjected to increase of temp.

L = length of the bar

d_1 = Diameter at bigger end

d_2 = Diameter at smaller end

t = Increase in temperature

α = Co-efficient of linear expansion.

We know that as a result of increase in temperature, the bar AB will tend to expand. But since it is fixed at both ends, therefore it will cause some compressive stress.

Due to rise in temperature

$$\delta l = l\alpha t$$

(1A) Let P = Load required to bring the deformed bar to the original length.

So, the Decrease in Length of the circular bar due to load 'P'.

$$\delta L = \frac{4PL}{\pi E d_1 d_2}$$

Now, $$L\alpha t = \frac{4PL}{\pi E d_1 d_2}$$

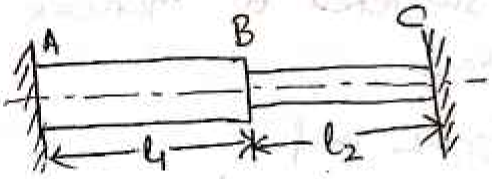
$$P = \frac{\pi E d_1 d_2 \alpha t}{4}$$

$$\sigma_{max} = \frac{P}{A} = \frac{P}{\frac{\pi}{4} d_2^2}$$

$$= \frac{\cancel{\pi} E d_1 \cancel{d_2} \alpha t}{\frac{\cancel{4}\cancel{\pi} d_2^{\cancel{2}}}{\cancel{4}}} = \frac{E d_1 \alpha t}{d_2}$$

$$\boxed{\sigma_{max} = \frac{E d_1 \alpha t}{d_2}}$$

Thermal Stress in Bars of Varying Section →



Consider a bar ABC fixed at its ends A & C and subjected to an increase of temperature.

Let l_1 = length of Portion AB

σ_1 = stress in portion AB

A_1 = C.S. area of Portion AB

l_2, σ_2, A_2 = corresponding values of Portion BC.

α = co-efficient of linear expansion.

t = Increase in temp.

The load shared by each portion is same

i.e $P_1 = P_2$

$$\sigma_1 A_1 = \sigma_2 A_2$$

Total deformation of the bar

$$\delta l = \delta l_1 + \delta l_2.$$

$$= \frac{\sigma_1 l_1}{E} + \frac{\sigma_2 l_2}{E}$$

$$= \frac{1}{E}\left[\sigma_1 l_1 + \sigma_2 l_2\right]$$

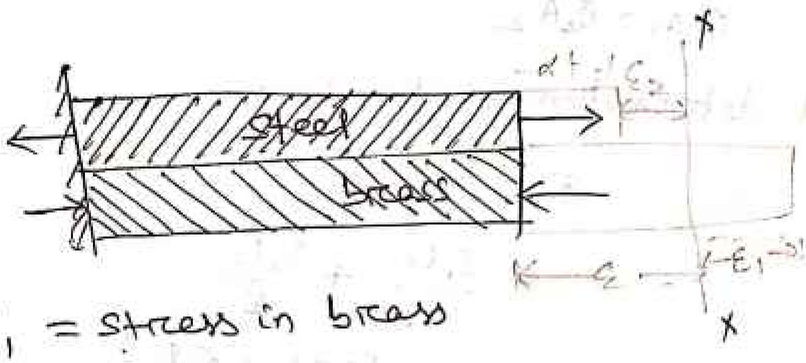
If the bars are of different material,

$$\delta l = \left[\frac{\sigma_1 l_1}{E_1} + \frac{\sigma_2 l_2}{E_2}\right].$$

Theremal Stresses in composite bars →

Whenever there is some increase or decrease in temp of a bar consisting of two or more different materials, it causes the bar to expand or contract. on account of different co-efficient of ~~exp~~ linear expansion, the two materials don't expand or contract by the same amount but expand or contract by different amounts.

Let's consider two bars (brass & steel) forms a composite bar. Let the bar be heated through some temperature. If the component members of the bar could have been free to expand, then no internal stresses would have induced. But since the two members are rigidly fixed, therefore the composite bar as a whole, will expand by the same amount. We know that the brass expands more than the steel ($\alpha_b > \alpha_s$). Therefore the free expansion of the brass will be more than that of steel. But, since both the members are not free to expand, therefore the expansion of the composite bar, as a whole, will be less than that of brass, but more than that of steel. It is thus obvious, that the brass will be subjected to compressive force, whereas the steel will be subjected to tensile force



let σ_1 = stress in brass

ε_1 = strain in brass

α_1 = co-efficient of linear expansion for brass

A_1 = cross-sectional area of brass bar.

$\sigma_2, \varepsilon_2, \alpha_2, A_2$ = corresponding values for steel.

ε = actual strain of the composite bar per unit length

As the compressive load on the brass is equal to the tensile load on the steel.

$$\sigma_1 A_1 = \sigma_2 A_2$$ (For equilibrium at X-X)

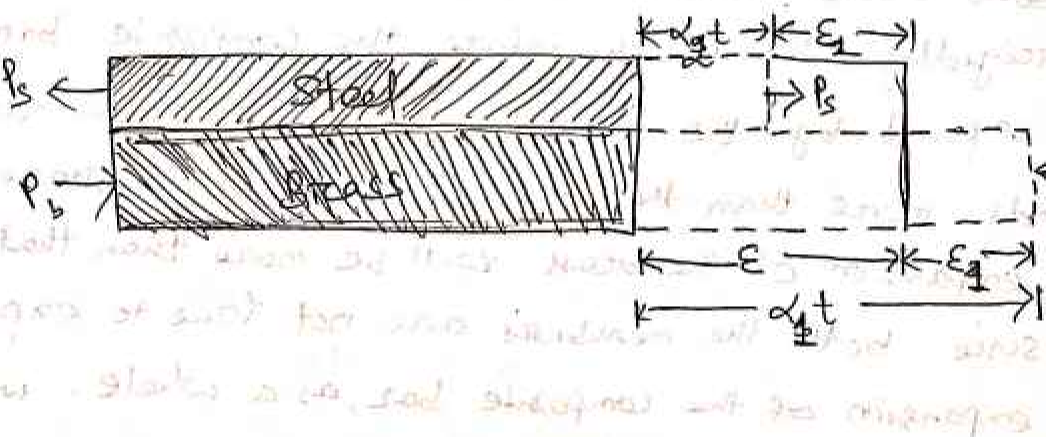
compressive strain in brass $\varepsilon_1 = \alpha_1 t - \varepsilon$ —— (1)

Tensile strain in steel $\varepsilon_2 = \varepsilon - \alpha_2 t$ —— (2)

Adding (1) & (2)

$$\varepsilon_1 + \varepsilon_2 = \alpha_1 t - \alpha_2 t$$

$$\boxed{\varepsilon_1 + \varepsilon_2 = t(\alpha_1 - \alpha_2)}$$

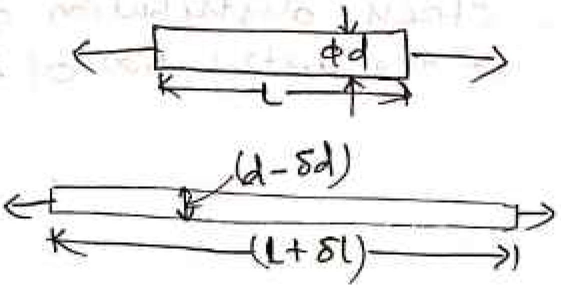


Saint Venant Principle →

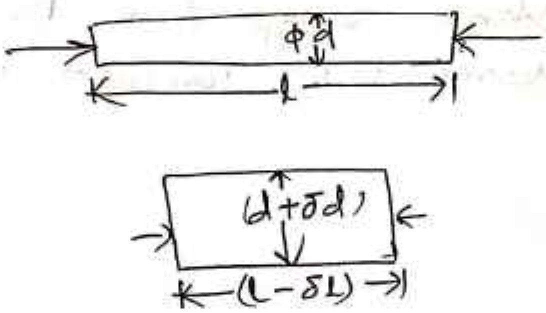
The actual distribution of load over the surface of its application will not affect the distribution of stress or strain on the section of the body which are appreciable distance away from the load. i.e. stress distribution is assume to be uniform irrespective of the distribution of load

Elastic constants →

If the bar is subjected to tensile load →



If the bar is subjected to Compressive load.

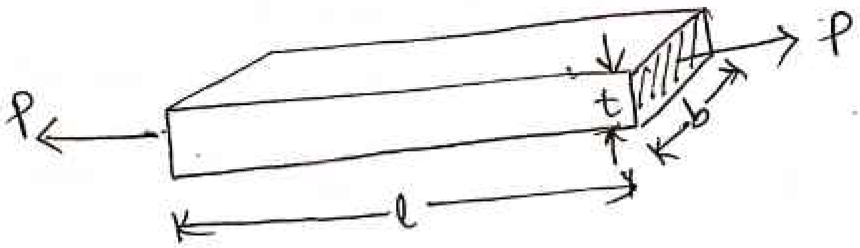


Generally there are 3-types of elastic constants

1) Young's modulus of elasticity
2) Rigidity Modulus
3) Bulk modulus

Volumetric strain = $\dfrac{\text{change in volume}}{\text{original volume}} = \dfrac{\Delta V}{V}$

Volumetric strain of a Rectangular body Subjected to an Axial Load →



Considering a bar, rectangular in section, subjected to an axial tensile force as shown in fig.

Let L = Length of the bar

b = Breadth of the bar

t = Thickness of the bar

P = Tensile force acting on bar

E = Modulus of elasticity

$\frac{1}{m}$ = Poisson's Ratio

$$\delta L = \frac{PL}{AE} = \frac{PL}{btE}$$

Linear $\sigma = \frac{P}{A} = \frac{P}{bt}$

linear strain $= \frac{\sigma}{E} = \frac{P}{btE}$ $\Rightarrow \frac{\delta l}{l} = \frac{P}{btE} \Rightarrow \delta l = \frac{PL}{btE}$

$$\frac{1}{m} = \frac{\text{Lateral Strain}}{\text{Linear Strain}}$$

Lateral Strain $= \frac{1}{m} \times$ linear strain

$= \frac{1}{m} \times \frac{P}{btE}$

Lateral strain $= \frac{\delta b}{b}$ or $\frac{\delta t}{t}$

$$\frac{\delta b}{b} = \frac{1}{m}\frac{P}{btE}$$

Change in breadth $\delta b = \frac{P}{mtE}$.

Change in ~~b~~ thickness $\delta t = \frac{P}{mbE}$

As a result of final tensile force,

Let the final length $= l + \delta l$.

final breadth $= b - \delta b$

final thickness $= t - \delta t$

Original volume of the body $V = lbt$

Final volume $= (l+\delta l)(b-\delta b)(t-\delta t)$

$= l\left(1+\frac{\delta l}{l}\right) \cdot b\left(1-\frac{\delta b}{b}\right) \cdot t\left(1-\frac{\delta t}{t}\right)$

$= lbt\left(1+\frac{\delta l}{l}\right)\left(1-\frac{\delta b}{b}\right)\left(1-\frac{\delta t}{t}\right)$

$= lbt\left[\left(1 - \frac{\delta b}{b} + \frac{\delta l}{l} - \frac{\delta l}{l}\cdot\frac{\delta b}{b}\right)\left(1-\frac{\delta t}{t}\right)\right]$

$= lbt\left[1 - \frac{\delta t}{t} - \frac{\delta b}{b} + \frac{\delta b}{b}\cdot\frac{\delta t}{t} + \frac{\delta l}{l} - \frac{\delta l}{l}\cdot\frac{\delta t}{t} - \frac{\delta l}{l}\cdot\frac{\delta b}{b} + \frac{\delta l}{l}\frac{\delta b}{b}\frac{\delta t}{t}\right]$

$= lbt\left[1 + \frac{\delta l}{l} - \frac{\delta b}{b} - \frac{\delta t}{t}\right]$ (neglecting smaller terms)

Change in volume $\delta V = \underset{\text{Final}}{\cancel{\text{original}}}$ volume − original volume (2)

$$= lbt\left(1+\frac{\delta l}{l}-\frac{\delta b}{b}-\frac{\delta t}{t}\right) - lbt$$

$$= lbt\left(\frac{\delta l}{l}-\frac{\delta b}{b}-\frac{\delta t}{t}\right)$$

$$= lbt\left[\frac{\frac{Pl}{btE}}{l} - \frac{\frac{P}{mtE}}{b} - \frac{\frac{P}{mbE}}{t}\right]$$

$$= lbt\left[\frac{P}{btE} - \frac{P}{mbtE} - \frac{P}{mbtE}\right]$$

$$= V \times \frac{P}{btE}\left[1 - \frac{1}{m} - \frac{1}{m}\right]$$

$$= V \times \frac{P}{btE}\left[1 - \frac{2}{m}\right]$$

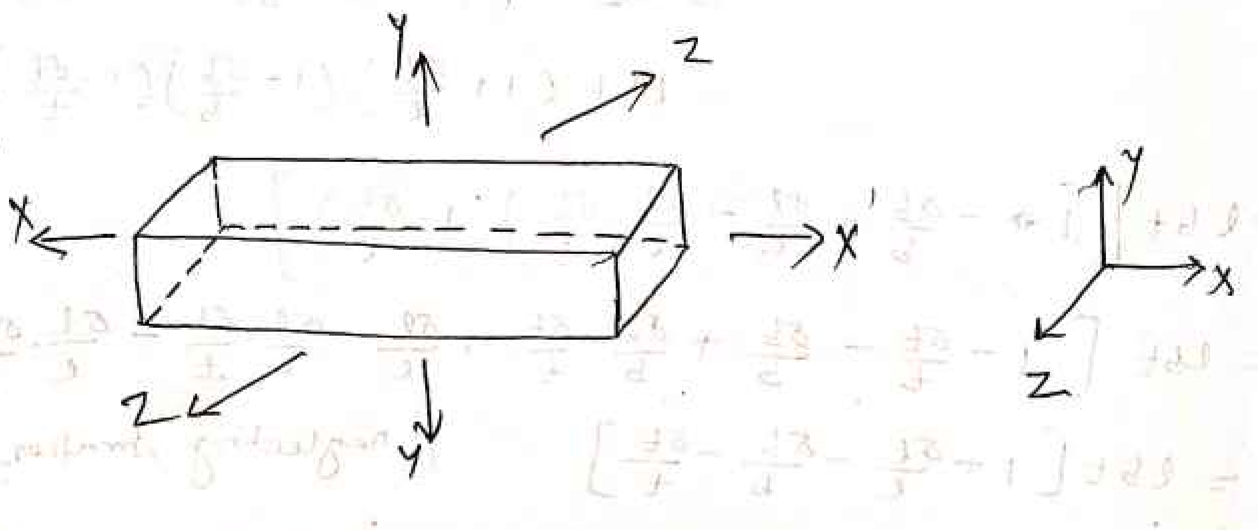
volumetric strain $\varepsilon_V = \frac{\delta V}{V}$

$$\frac{\delta V}{V} = \frac{\cancel{V}\frac{P}{btE}\left[1-\frac{2}{m}\right]}{\cancel{V}}$$

$$= \frac{P}{btE}\left[1-\frac{2}{m}\right]$$

$$\boxed{\varepsilon_V = \varepsilon_l\left[1-\frac{2}{m}\right]} \qquad \left(\because \varepsilon_l = \frac{P}{btE}\right)$$

Volumetric strain of a Rectangular Body subjected to three Mutually Perpendicular forces →



Consider a rectangular body subjected to direct tensile stresses along three mutually perpendicular axes.

let σ_x = Stress in x-direction.

σ_y = Stress in y-direction

σ_z = Stress in z-direction.

E = Young's Modulus of elasticity

Strain in x-direction due to σ_x only, $\varepsilon_x = \frac{\sigma_x}{E}$

Similarly, $\varepsilon_y = \frac{\sigma_y}{E}$

$$\varepsilon_z = \frac{\sigma_z}{E}$$

Resulting strain in x-dirn due to all the stress.

$$\varepsilon_x = \varepsilon_{lx} - \varepsilon_{ly} - \varepsilon_{lz}$$

$$= \frac{\sigma_x}{E} - \frac{\sigma_y}{mE} - \frac{\sigma_z}{mE}$$

Because, the resulting strains in three directions, may be found out by the Principle of superposition i.e. by adding algebraically the strains in each direction due to each individual stress.

$$\varepsilon_y = \varepsilon_{ly} - \varepsilon_{lx} - \varepsilon_{lz}$$

$$= \frac{\sigma_y}{E} - \frac{\sigma_x}{mE} - \frac{\sigma_z}{mE}$$

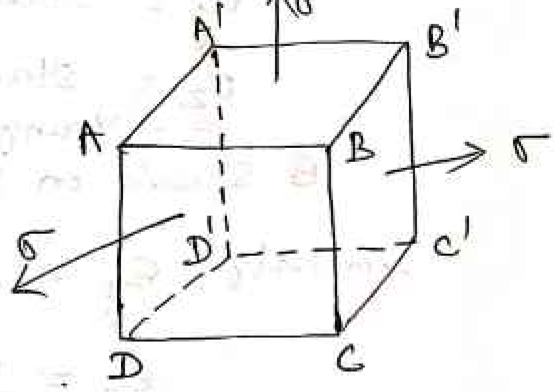
$$\varepsilon_z = \varepsilon_{lz} - \varepsilon_{lx} - \varepsilon_{ly}$$

$$= \frac{\sigma_z}{E} - \frac{\sigma_x}{mE} - \frac{\sigma_y}{mE}$$

volumetric strain $\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$

Relation Between Bulk Modulus & Young's Modulus →

Let's consider a cube ABCD A'B'C'D' is subjected to three mutually Perpendicular tensile stresses of equal intensity.



Let l = Length of the cube
E = Young's modulus for the material of the block.

Now consider the deformation of one side of cube (say AB) under the action of three mutually ⊥ar stresses.

1) Tensile strain equal to $\frac{\sigma}{E}$ due to stresses on face BB'C & AA' DD'.

2) compressive lateral strain equal to $\frac{1}{m}\frac{\sigma}{E}$ due to stresses on faces AA'B'B & DD'C'C.

3) compressive Lateral Strain equal to $\frac{\sigma}{mE}$ due to stress on face ABCD & A'B'C'D'.

Therefore net tensile strain, the side AB will suffer, due to these stresses.

$$\frac{\delta l}{l} = \frac{\sigma}{E} - \frac{\sigma}{mE} - \frac{\sigma}{mE}$$

$$\varepsilon_l = \frac{\sigma}{E}\left(1 - \frac{2}{m}\right)$$

Original volume of the cube $V = l^3$

$$\frac{\delta V}{\delta l} = 3l^2$$

$$\delta V = 3l^2 \, \delta l$$
$$= 3l^3 \frac{\delta l}{l}$$
$$= 3V \cdot \varepsilon_l$$

$$\delta V = 3V \times \frac{\sigma}{E}\left(1-\frac{2}{m}\right)$$

$$\frac{\delta V}{V} = \frac{3\sigma}{E}\left(1-\frac{2}{m}\right)$$

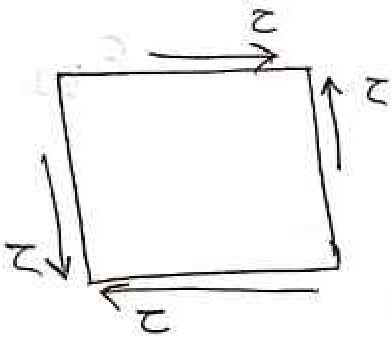
we know, $K = \dfrac{\sigma}{\left(\frac{\delta V}{V}\right)}$

$$K = \frac{\sigma}{\frac{3\sigma}{E}\left(1-\frac{2}{m}\right)} = \frac{E}{3\left(1-\frac{2}{m}\right)}$$

$$\boxed{K = \frac{mE}{3(m-2)}}$$

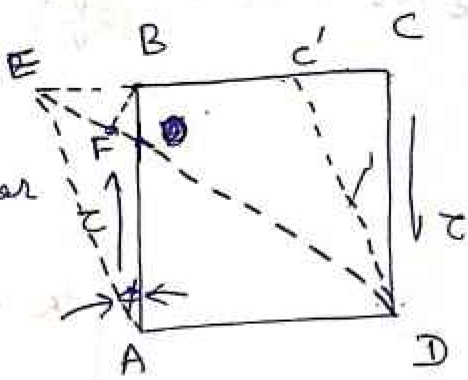
Principle of shear stress →

A shear stress across a plane, is always accompanied by a balancing shear stress across the plane & normal to it. i.e. Shear stress always exist in pairs.



Relationship between Modulus of Elasticity and modulus of Rigidity.

Consider a square element of sides 'a' subjected to pure shear 'τ'. AEC'D is the deformed shape due to shear 'τ'. Drop a perpendicular BF to diagonal ED. Let 'φ' be the shear strain & 'G' is the modulus of rigidity.



Strain in diagonal BD $= \dfrac{DE - BD}{BD}$

Since B is closed to F. So, the point B & F are assumed to be same.

$$\varepsilon_{BD} = \frac{DE - DF}{DF}$$

$\frac{AB}{BD} = \cos 45°$, $BD = \sqrt{2} \cdot$ ...

$$= \frac{EF}{DF} = \frac{BE \cos 45°}{DF} = \frac{BE}{\sqrt{2}\,DF} = \frac{BE}{\sqrt{2} \times (\sqrt{2} \times AB)}$$

$$= \frac{BE}{2AB} = \frac{\tan\phi}{2} = \frac{\phi}{2} \quad (\because \phi \simeq \tan\phi)$$

$$\text{or } G = \frac{\tau}{\phi}$$

$$G = \frac{\tau}{\phi} \Rightarrow \phi = \frac{\tau}{G}$$

$$\varepsilon_{BD} = \frac{\phi}{2} = \frac{\tau}{2G}$$

Linear strain of the diagonal BD is half of the shear strain ($\varepsilon_{BD} = \frac{\phi}{2}$) & is Tensile in nature. Similarly it can be proved that the linear strain of the diagonal AC is also equal to half of the shear strain but compressive in nature.

The Tensile strain on diagonal BD due to tensile stress

$$= \frac{\tau}{E}$$

Tensile strain on diagonal BD due to compressive stress on diagonal AC, $= \frac{\tau}{mE}$

The Combined effect of the above two stresses on diagonal BD.

$$= \frac{\tau}{E} + \frac{1}{m} \times \frac{\tau}{E}$$

$$= \frac{\tau}{E}\left(1+\frac{1}{m}\right)$$

($\because$ net strain

$$\varepsilon = \frac{\tau}{E} - \frac{\tau}{mE}$$

$$= \frac{\tau}{E} - \frac{(-\tau)}{mE}$$

$$= \frac{\tau}{E} + \frac{\tau}{mE}$$)

$-\tau \rightarrow$ comp. stress

$$\frac{\phi}{2} = \frac{\tau}{E}\left(1+\frac{1}{m}\right)$$

$$\frac{\tau}{2G} = \frac{\tau}{E}\left(1+\frac{1}{m}\right)$$

$$G = \frac{E}{2\left(1+\frac{1}{m}\right)} = \frac{mE}{2(m+1)}$$

$$\boxed{G = \frac{mE}{2(m+1)}}$$

(Note: The effect of shear stresses on sides of cube causes tensile stress on diagonal BD & compressive stresses on AC)

- **Relationship among E K & G →**

we know,

$$K = \frac{mE}{3(m-2)} = \frac{E}{3(1-2\mu)} \quad -①$$

$$G = \frac{mE}{2(m+1)} = \frac{E}{2(1+\mu)} \quad -②$$

$$K = \frac{mE}{3(m-2)} = \frac{E}{3(1-2\mu)}$$

$$3K(1-2\mu) = E$$

$$1 - 2\mu = \frac{E}{3K}$$

$$\mu = \frac{1}{2}\left(1 - \frac{E}{3K}\right)$$

Now putting the value of μ in eqn (2)

$$G = \frac{E}{2(1+\mu)}$$

$$= \frac{E}{2\left(1 + \frac{1}{2} - \frac{E}{6K}\right)}$$

$$= \frac{E}{2\left(\frac{3}{2} - \frac{E}{6K}\right)} = \frac{E}{3 - \frac{E}{3K}}$$

$$= \frac{3KE}{9K - E}$$

$$G(9K - E) = 3EK$$

$$9GK - GE = 3EK$$

$$3EK + GE = 9GK$$

$$E(3K + G) = 9GK$$

$$\boxed{E = \frac{9GK}{3K+G}}$$

Strain energy →

When a body is subjected to external load, the body will undergo some deformation & it absorbed some energy. If the body is stretched within elastic limit it stores energy & released the same on the removal of the load i.e it will spring back to its original position. This energy which is absorbed in a body, when strained within its elastic limit, is known as strain energy.

It has been experimentally seen that this strain-energy is always capable of doing some work. The amount of strain energy, in a body is found out by the principal of work. Mathematically, strain energy = work done.

Resilience →(U)

Resilience is nothing but strain energy i.e. energy stored within the elastic limit at any point is known as Resilience.

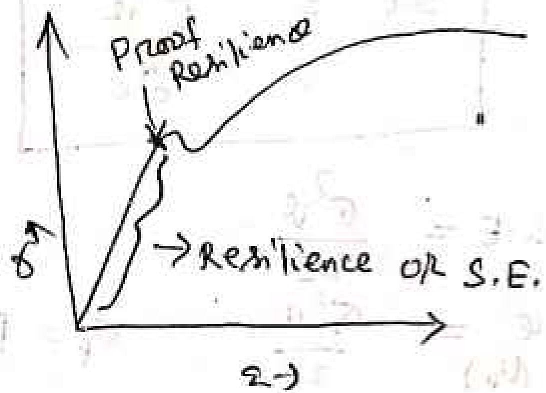
Proof Resilience →(U_p)

The maximum strain energy stored in a body is known as Proof Resilience. i.e. The strain energy stored at elastic limit when the body is stressed.

The corresponding stress is known as proof stress.

Modulus of Resilience →(U_p/V)

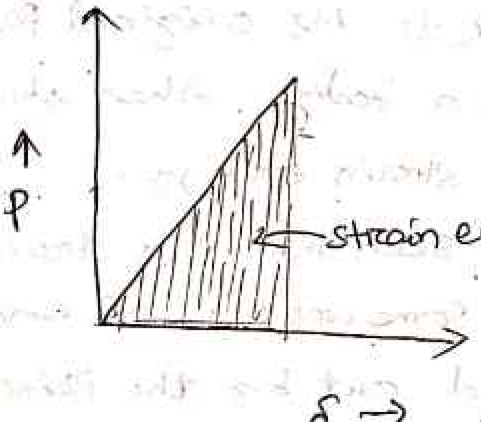
The Proof Resilience per unit volume of the material is known as Modulus of Resilience.



Types of Loading →

1) Gradually applied load
2) Suddenly applied Load
3) Impact load.

*



Let's consider a bar of cross-sectional area A, length 'L' is subjected to an axial load of P.

Let stress developed = σ

corresponding strain = ε

Young's modulus of elasticity = E

Work done by the load applied = $\frac{1}{2} P \times \delta$

When deformation is zero, R = 0
when deformation is δ, R = P } ∵ R = Resisting force

Strain energy (due to Resisting force)

$$S.E = \frac{1}{2} R \cdot \delta$$

$$= \frac{1}{2} \times \sigma A \cdot \frac{\sigma}{E} L$$

$$= \frac{1}{2} \frac{\sigma^2}{E} (AL)$$

$\sigma = \frac{R}{A}$

$R = \sigma A$

$E = \frac{\sigma}{\varepsilon}$

$\varepsilon = \frac{\sigma}{E}$

$\frac{\delta}{L} = \frac{\sigma}{E}$

$\delta = \frac{\sigma L}{E}$

$$\boxed{S.E = \frac{\sigma^2 V}{2E}}$$

Resilience (U) = S.E = $\frac{\sigma^2 V}{2E}$

Proof Resilience (U_P) = $\frac{\sigma_p^2 V}{2E}$ σ_p = Proof stress

Modulus of Resilience $= \frac{U_p}{V} = \frac{\sigma_p^2 \cancel{V}}{2E \times \cancel{V}} = \frac{\sigma_p^2}{2E}$

Strain energy stored in a body when the Load is Gradually applied →

On gradually applied load, Loading starts from zero and ~~g~~ increases gradually till the body is fully loaded.

{e.g. = crane., when we lower a body with the help of a crane, the body first touches the platform on which it is to be placed. on further releasing the chain, the platform goes on loading till it is fully loaded ~~the~~ by the body.)

Let load starts from 0 to P

extension 0 to δ.

So, work done = $P_{average}$ × displacement

$$W.D = \frac{P}{2} \times \delta = \frac{1}{2} P \times \delta$$

$$\text{strain energy} = \frac{1}{2} R \cdot \delta$$

$$S.E = \frac{1}{2} P \cdot \delta = \frac{1}{2} \sigma A \delta$$

~~For equilibrium~~, $S.E = W.D$

$$\cancel{\tfrac{1}{2}} \sigma A \cancel{\delta} = \cancel{\tfrac{1}{2}} P \cancel{\delta}$$

$$\boxed{\sigma = \frac{P}{A}}$$

$$\boxed{\sigma_{gradually} = \frac{P}{A}}$$

(30) Strain energy stored in a body when the load is suddenly appl[ied]

Sometimes, in factories & workshops, the load is suddenly applied on a body e.g. when we lower a body with the help of a crane, the body is first of all, just above the platform on which it is to be placed. If the chain breaks at once at this moment the whole load of the body begins to act on the platform. This is the case of suddenly applied load.

Let P = load applied

A = C.S area of the bar

l = Length of the bar

E = Modulus of elasticity

δl = Deformation of the bar

σ = Stress produced

Since the load is applied suddenly, 'P' is constant throughout the process of deformation of the bar.

Work done = Force x distance

$= P \times \delta l$

Strain energy stored $'U' = \frac{\sigma^2}{2E} \times Al$

$S.E = W.D$

$\frac{\sigma^2}{2E} \times Al = P \times \delta l$

$= P \times \frac{\sigma . l}{E}$

$\left(\sigma = E . \varepsilon = E . \frac{\delta l}{l} \quad \delta l = \frac{\sigma l}{E} \right)$

$\frac{\sigma^{\cancel{2}} A \cancel{l}}{2 \cancel{E}} = P \frac{\cancel{\sigma} \cancel{l}}{\cancel{E}}$

$\sigma = \frac{2P}{A}$

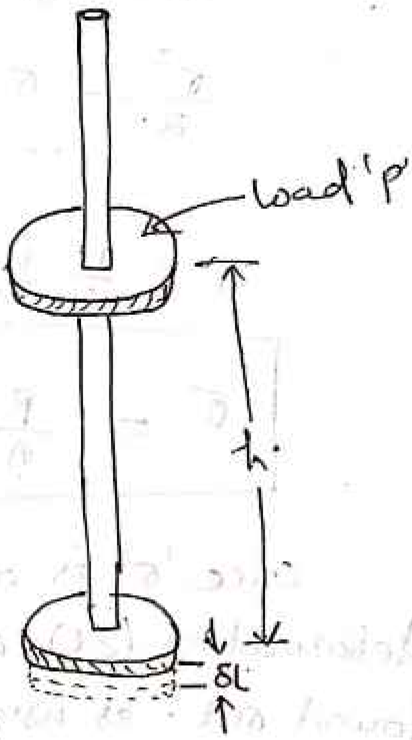
$\boxed{\sigma_{suddenly} = 2 \times \frac{P}{A}}$

(31)

Strain energy stored in a body when the Load is applied with Impact →

Sometimes in factories and workshops, the load with impact is applied on a body e.g. when we lower a body with the help of a crane, and the chain breaks while the load is being lowered, the load falls through a distance, before it touches the platform. This is the case of a load applied with impact.

Let's consider a bar is subjected to load applied with impact as shown in fig.



Let P = load applied with impact,

A = C.S. area of the bar

E = Modulus of elasticity of the bare material.

L = Length of the bar

δl = Deformation of the bar as a result of this Load.

σ = Stress induced by the application of this Load with impact.

h = Height through which the load will fall, before impact on collar of the bar

work done = Load × distance

$$= P(h+\delta l)$$

energy stored, $U = \frac{\sigma^2}{2E} v$

$$= \frac{\sigma^2}{2E}(Al)$$

Since energy stored = work done

$$\frac{\sigma^2}{2E} Al = P(h+\delta l)$$

$$\frac{\sigma^2}{2E} Al = Ph + P\delta l$$

$$= Ph + P \times \frac{\sigma}{E} l$$

$(\because E = \frac{\sigma}{\varepsilon}$

$\varepsilon = \frac{\sigma}{E}$

$\frac{\delta l}{l} = \frac{\sigma}{E}$

$\delta l = \frac{\sigma}{E} l)$

$$\frac{\sigma^2}{\cancel{2}} \frac{(Al)}{2E} - \sigma\left(\frac{Pl}{E}\right) - Ph = 0$$

Multiplying $\frac{E}{Al}$ throughout the eq^n

$$\sigma^2 \left(\frac{Al}{2E}\right) \times \frac{E}{Al} - \sigma\left(\frac{Pl}{E}\right) \times \frac{E}{Al} - Ph \times \frac{E}{Al} = 0$$

$$\frac{\sigma^2}{2} - \sigma\left(\frac{P}{A}\right) - \frac{PEh}{Al} = 0$$

$ax^2+bx+c=0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$\sigma = \frac{P}{A} \pm \sqrt{\left(\frac{-P}{A}\right)^2 - 4 \times \frac{1}{2} \times \left(-\frac{PEh}{Al}\right)}$$

$$\boxed{\sigma = \frac{P}{A}\left[1 \pm \sqrt{1 + \frac{2AEh}{Pl}}\right]}$$

Once 'σ' is obtained, the corresponding instantaneous deformation (δl) or the strain energy stored may be found out as usual.

Corollary :-

when δl is very very small.

$$\text{workdone} = P(h + \delta l)$$

$$= Ph$$

$$\frac{\sigma^2}{2E} Al = Ph$$

$$\sigma^2 = \frac{2EPh}{Al}$$

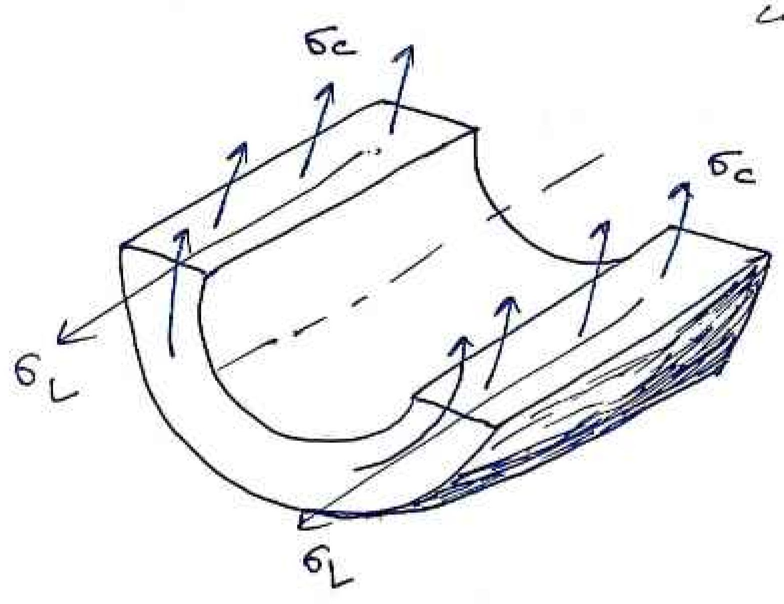
$$\boxed{\sigma = \sqrt{\frac{2EPh}{Al}}}$$

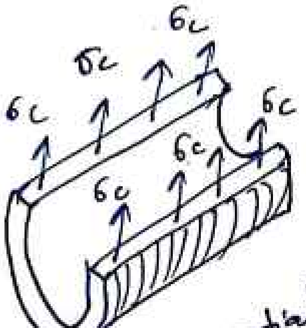
Thin cylindrical and spherical shells

A cylindrical vessel or shell may be thin or thick depending upon the thickness of the plate in relation to the internal diameter of the cylinder.

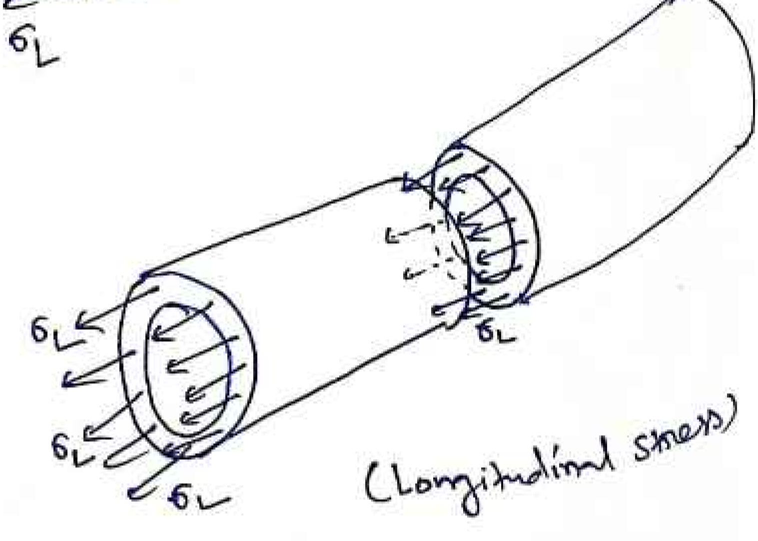
Basis	Thin cylinder	Thick cylinder
According to thickness to diameter ratio	$\frac{t}{d} \leq \frac{1}{20}$	$\frac{t}{d} > \frac{1}{20}$
According to internal fluid Pressure to the Strength of cylinder	$\frac{P_i}{\sigma_t} < \frac{1}{6}$	$\frac{P_i}{\sigma_t} > \frac{1}{6}$

where σ_t = Tensile strength of cylinder material





(circumferential stress)



(Longitudinal stress)

When the cylinders are subjected to internal fluid pressure the following types of stresses are developed.

1) Hoop and circumferential stresses.

2) Longitudinal stresses.

Note: Compared to hoop & longitudinal stresses, one more stress called radial stress whose value is very small. So it can be neglected.

Hoop ~~and~~ or Circumferential Stress →

Let d = diameter of the cylinder

t = Thickness of the cylinder

p = interenal flued Pressure

σ_c = circumferential stress

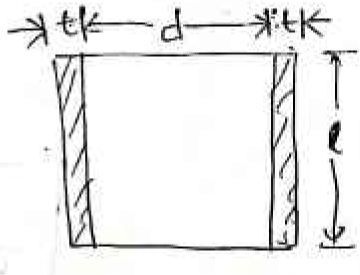
Brusting force due to internal fluid Pressure,

$$F_B = p \times (d \times l)$$

Resisting force due to σ_c,

$$\therefore F_R = \sigma_c \times (l \times t + l \times t)$$

$$= 2lt\,\sigma_c$$



For equillibrium, $F_B = F_R$

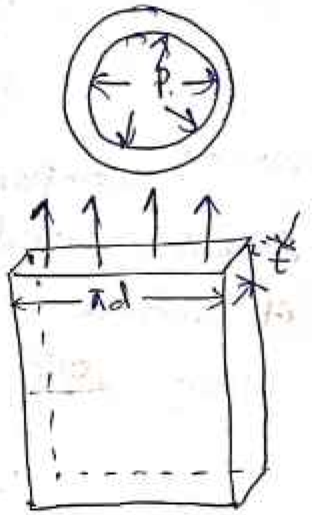
$$p\,d\cancel{l} = 2\cancel{l}t\,\sigma_c$$

$$\boxed{\sigma_c = \frac{pd}{2t}}$$

Longitudinal stress →

Bursting force due to 'p' = Pressure × Area

$$= p\left(\frac{\pi}{4}d^2\right)$$



Resisting force due to σ_L,

$$= \sigma_L \times (\pi d t)$$

For equillibrium.

$$\sigma_L(\cancel{\pi}\cancel{d}t) = p\left(\frac{\cancel{\pi}}{4}d^{\cancel{2}}\right)$$

$$\boxed{\sigma_L = \frac{pd}{4t}}$$

Maximum shear stress →

$$\tau_{max} = \frac{\sigma_c - \sigma_L}{2}$$

$$= \frac{\frac{pd}{2t} - \frac{pd}{4t}}{2}$$

$$= \frac{pd}{8t}$$

$$\boxed{\tau_{max} = \frac{pd}{8t}}$$

In cylindrical shell, at any point on its circumference, there is a set of two mutually perpendicular stresses σ_c and σ_L which are principal stresses and as such the planes in which these act are the principal planes.

Change in dimensions of a thin cylindrical shell due to Internal Pressure →

In a cylindrical shell subjected to an internal pressure, its wall will also be subjected to lateral strain. The effect of the lateral strains is to ~~be~~ cause some change in the dimensions of the shell.

Let's consider a thin cylinder,

p = internal fluid pressure

l = length of the cylinder

d = diameter of the cylinder

t = Thickness of the shell.

We know, $\sigma_c = \frac{pd}{2t}$, $\sigma_l = \frac{pd}{4t}$

Now let δd = change in diameter of the shell

δl = change in length of the shell

$\frac{1}{m} = \mu$ = Poisson's ratio.

Circumferential strain, $\varepsilon_1 = \frac{\delta d}{d} = \frac{\sigma_c}{E} - \frac{\sigma_l}{mE}$

$$= \frac{pd}{2tE} - \frac{pd}{4tmE}$$

$$= \frac{pd}{2tE}\left[1 - \frac{1}{2m}\right] = \frac{pd}{2tE}\left[1 - \frac{\mu}{2}\right]$$

$$\boxed{\delta d = \frac{pd^2}{2tE}\left[1 - \frac{1}{2m}\right]}$$

Longitudinal strain, $\varepsilon_2 = \frac{\delta l}{l} = \frac{\sigma_l}{E} - \frac{\sigma_c}{mE}$

$$= \frac{pd}{4tE} - \frac{pd}{2tmE} = \frac{pd}{2tE}\left[\frac{1}{2} - \frac{1}{m}\right]$$

$$\boxed{\delta l = \frac{pdl}{2tE}\left[\frac{1}{2} - \frac{1}{m}\right]}$$

Change in volume of a thin cylinderical shell due to internal Pressure →

Let l = original length

d = original diameter

δl = Change in length due to Pressure

δd = change in diameter due to pressure

original volume $V = \frac{\pi}{4} d^2 l$

Final volume $V' = \frac{\pi}{4}(d+\delta d)^2 (l+\delta l)$

$$= \frac{\pi}{4}\left(d^2 + \cancel{\delta d^2}^{\,0} + 2d\delta d\right).(l+\delta l)$$

$$= \frac{\pi}{4}\left(d^2 l + d^2 \delta l + 2d\delta d \cdot l + \cancel{2d\,\delta d\,\delta l}^{\,0}\right)$$

$$= \frac{\pi}{4} d^2 l + \frac{\pi}{4} d^2 \delta l + \frac{\pi}{4} \times 2d\delta d \cdot l$$

Change in volume = $V' - V$

$$\Delta V = \cancel{\frac{\pi}{4} d^2 l} + \frac{\pi}{4} d^2 \delta l + \frac{\pi}{4} \times 2d\delta d \cdot l - \cancel{\frac{\pi}{4} d^2 l}$$

$$= \frac{\pi}{4}\left(d^2 \delta l + 2dl \cdot \delta d\right)$$

Volumetric strain $= \frac{\Delta V}{V} = \frac{\cancel{\frac{\pi}{4}}(d^2\delta l + 2dl\delta d)}{\cancel{\frac{\pi}{4}} d^2 l}$

$$= \frac{\cancel{d^2}\delta l}{\cancel{d^2} l} + \frac{2\cancel{d}\cancel{l}\delta d}{d^{\cancel{2}}\cancel{l}}$$

$$= \frac{\delta l}{l} + 2 \cdot \frac{\delta d}{d}$$

$$= \varepsilon_l + 2\varepsilon_c$$

$$\left[\frac{\Delta V}{V} = \varepsilon_l + 2\varepsilon_c\right]$$

now ε_l = Longitudinal strain

ε_c = Circumferential strain.

$$\boxed{\Delta V = V(\varepsilon_l + 2\varepsilon_c)}$$

Thin Spherical Shell →

Consider a thin spherical shell subjected to an internal pressure.

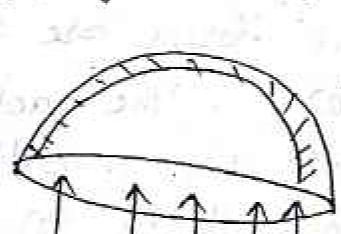
let p = Intensity of internal pressure

d = Diameter of shell

t = Thickness of the shell.

As a result of internal pressure, the shell is likely to be torn away along the centre of the sphere.

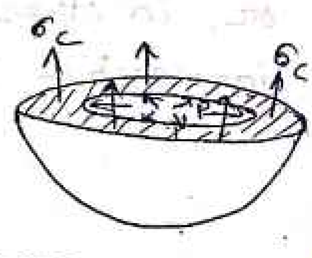
Therefore, Brusting force acting along the centre of sphere $= p \times \frac{\pi}{4} d^2$



Stress in the shell material,

$$\sigma = \frac{\text{Total Pressure}}{\text{Resisting section}}$$

$$= \frac{p \times \frac{\pi}{4} d^2}{\pi d t}$$



Or, Brusting force = Resisting force

$$p \times \frac{\pi}{4} d^2 = \sigma \pi d t$$

$$\boxed{\sigma = \frac{pd}{4t}}$$

Factor of safety →

It is defined as the ratio of maximum or ultimate stress to the working stress.

$$\text{Factor of safety} = \frac{\text{ultimate stress}}{\text{working stress}}$$

Saint venant's Principle →

If the forces are acting on a small portion of the surface of an elastic body are replaced by another statically equivalent system of forces acting on the same portion of the surface, this redistribution of loading produces substantial changes in the stresses locally but has a negligible effect on the stresses at distances which are large in comparision with the linear dimensions of the surface on which the forces are changed.

or, The actual distribution of load over the surface of its application will not affect the distribution of stress or strain on the section of the body which are in appreciable distance away from the load.

or, in other words, stress distribution is assume to be uniform irrespective of the distribution of load.

PRINCIPAL STRESSES AND PRINCIPAL STRAINS

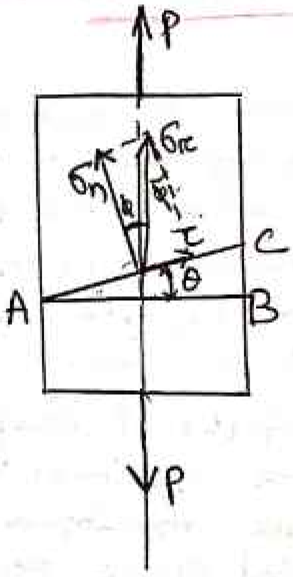
At any point within a stressed body, no matter how complex the state stress may be, there always exist three mutually perpendicular planes, on each of which the resultant stress is a normal stress. These mutually perpendicular planes are called principal planes and the resultant normal stresses acting on them are called principal stresses.

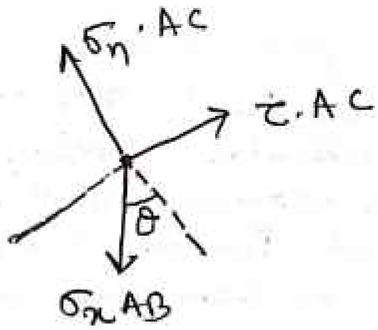
In case of 2-D Problems, one of the principal stress is zero & out of other two, one will be maximum & other will be minimum value. The plane where maximum principal stress is acting, called major principal plane and other one is called minor principal plane.

OR, It has been observed that at any point in a strained material, there are three planes, mutually perpendicular to each other which carry direct stresses only & no shear stress. It may be noted that out of these three direct stresses, one will be maximum & other will be minimum. These perpendicular planes which have no shear stress are known as principal planes & the direct stresses along these planes are known as principal stresses.

Note: The planes on which the maximum shear stress act are known as planes of maximum shear.

Stresses on an oblique plane under uniaxial loading :-





Consider a bar of C-S. area 'A' & carrying a load 'P' applied along the axis of the bar. On a plane AB perpendicular to the line of action of load 'P', &

The direct stress $\sigma_x = \frac{P}{A}$

Let AC be an oblique plane inclined at an angle 'θ' to the plane AB. Let σ_n = Normal stress & τ = shear stress on the plane AC. Taking thickness of the bar is unity & Resolving the forces ⊥ar to the plane AC,

$$\sigma_n \cdot AC = \sigma_x \cdot AB \cos\theta$$

$$\sigma_n = \sigma_x \cdot \frac{AB}{AC} \cos\theta$$

$$= \sigma_x \cdot \cos\theta \cdot \cos\theta$$

$$= \sigma_x \cos^2\theta$$

$$= \sigma_x \left(\frac{1+\cos 2\theta}{2}\right) \quad \left(\cos 2\theta = 2\cos^2\theta - 1,\ \cos^2\theta = \frac{1+\cos 2\theta}{2}\right)$$

$$\boxed{\sigma_n = \frac{\sigma_x}{2}(1+\cos 2\theta)}$$

when $\theta = 0^\circ$, $\sigma_n = \frac{\sigma_x}{2}(1+1) = \sigma_x$ (Maximum)

$\theta = 90^\circ$, $\sigma_n = \frac{\sigma_x}{2}(1-1) = 0$ (Minimum)

∴ Now Resolving the forces parallel to the plane AC,

$$\tau \cdot AC = \sigma_n AB \sin\theta$$

$$\tau = \sigma_n \cdot \frac{AB}{AC} \cdot \sin\theta$$

$$= \sigma_x \cdot \cos\theta \cdot \sin\theta$$

$$\boxed{\tau = \frac{\sigma_x}{2} \sin 2\theta}$$

Resultant stress, $\sigma_R = \sqrt{\sigma_n^2 + \tau^2}$

$$= \sqrt{\left\{\frac{\sigma_x}{2}(1+\cos 2\theta)\right\}^2 + \left(\frac{\sigma_x}{2}\sin 2\theta\right)^2}$$

$$= \frac{\sigma_x}{2}\sqrt{(1+\cos 2\theta)^2 + (\sin 2\theta)^2}$$

$$= \frac{\sigma_x}{2}\sqrt{1+\cos^2 2\theta + 2\cos 2\theta + \sin^2 2\theta}$$

$$= \frac{\sigma_x}{2}\sqrt{2(1+\cos 2\theta)}$$

$$= \frac{\sigma_x}{2}\sqrt{2 \times 2\cos^2\theta} \quad \left(\because \cos 2\theta = 2\cos^2\theta - 1,\ 2\cos^2\theta = 1+\cos 2\theta\right)$$

$$= \frac{\sigma_x}{2} \cdot 2\cos\theta$$

$$\boxed{\sigma_R = \sigma_x \cos\theta}$$

If ϕ = Angle bet^n σ_n & σ_R.

$$\tan\phi = \frac{\tau}{\sigma_n} = \frac{\frac{\sigma_x}{2}\sin 2\theta}{\frac{\sigma_x}{2}(1+\cos 2\theta)} = \frac{2\sin\theta\cdot\cos\theta}{1+2\cos^2\theta - 1}$$

$$= \frac{2\sin\theta\cdot\cos\theta}{2\cos^2\theta} = \tan\theta$$

$$\tan\phi = \tan\theta$$

$$\boxed{\phi = \theta}$$

$\tau = \frac{\sigma_x}{2}\sin 2\theta$, Shear Stress is maximum when $\sin 2\theta$ is maximum

i.e. $\sin 2\theta = \pm 1$

$2\theta = 90^\circ$ or 270°

$\theta = 45^\circ$ or 135°

$$\tau = \frac{\sigma_x}{2}\sin(2\times 45^\circ)$$

$$\boxed{\tau = \frac{\sigma_x}{2}}$$

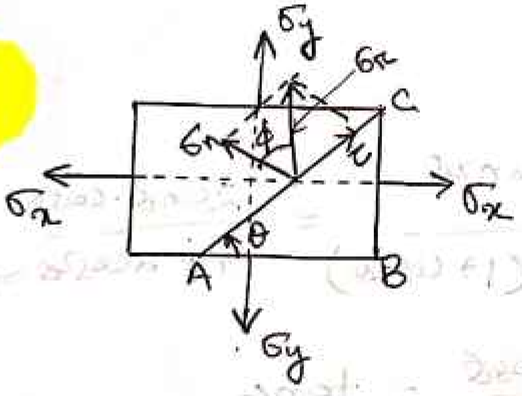
Therefore for a state of uniaxial stress the maximum tangential stress occurs along planes, the normals to which makes angle of 45° & 135° with the direction of the Load.

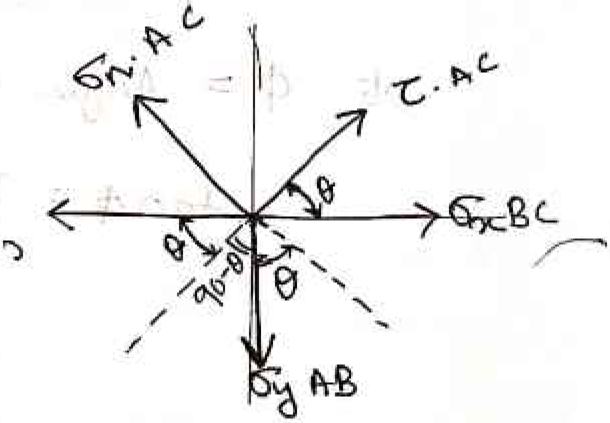
Note: From this result a very important conclusion follows that if a material is such that its shear strength is less than half of its tensile strength, then the material will fail by shear when subjected to uniaxial tensile stress.

Stresses on an oblique Plane under Biaxial Loading

Case-1: Like stress

Let's consider a bar is subjected to biaxial stresses. Let both are tensile in nature (σ_x & σ_y)





Let AC be the oblique plane inclined at an angle θ with the plane AB which is parallel to the line of action of σ_x stress. Let σ_n & τ be the normal & shear stresses on the planes AC. The forces acting at any point on AC

Resolving the forces ⊥are to the Plane AC, we get

$$\sigma_n AC = \sigma_x \cdot BC \cdot \sin\theta + \sigma_y \cdot AB \cdot \cos\theta$$

$$\sigma_n = \sigma_x \cdot \frac{BC}{AC} \cdot \sin\theta + \sigma_y \cdot \frac{AB}{AC} \cdot \cos\theta$$

$$= \sigma_x \cdot \sin\theta \cdot \sin\theta + \sigma_y \cdot \cos\theta \cdot \cos\theta$$

$$\boxed{\sigma_n = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta}$$

$$= \sigma_x \left(\frac{1-\cos 2\theta}{2}\right) + \sigma_y \left(\frac{1+\cos 2\theta}{2}\right)$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x \cos 2\theta}{2} + \frac{\sigma_y}{2} + \frac{\sigma_y \cos 2\theta}{2}$$

$$\boxed{\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_y - \sigma_x}{2}\right)\cos 2\theta}$$

Similarly, resolving forces parallel to the plane AC,

$$\tau \cdot AC = -\sigma_x \cdot BC \cdot \cos\theta + \sigma_y \cdot AB \sin\theta$$

$$\tau = -\sigma_x \cdot \frac{BC}{AC} \cos\theta + \sigma_y \cdot \frac{AB}{AC} \cdot \sin\theta$$

$$= -\sigma_x \cdot \sin\theta \cdot \cos\theta + \sigma_y \cdot \cos\theta \cdot \sin\theta$$

$$= \left(-\sigma_x + \sigma_y\right) \frac{2\sin\theta \cdot \cos\theta}{2}$$

$$\boxed{\tau = \frac{1}{2}(\sigma_y - \sigma_x) \sin 2\theta}$$

The shear stress is maximum when $\theta = 45^\circ$ & value is

$$\tau_{max} = \frac{\sigma_y - \sigma_x}{2} \sin(2 \times 45^\circ)$$

$$\boxed{\tau_{max} = \left(\frac{\sigma_y - \sigma_x}{2}\right)}$$

The corresponding value of $\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_y - \sigma_x}{2} \cos 2 \times 45^\circ$

$$\boxed{\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right)}$$

Resultant stress, $\sigma_R = \sqrt{\sigma_n^2 + \tau^2}$

$$= \sqrt{\left\{\left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_y-\sigma_x}{2}\right)\cos 2\theta\right\}^2 + \left\{\left(\frac{\sigma_y-\sigma_x}{2}\right)\sin 2\theta\right\}^2}$$

$$= \sqrt{\left(\frac{\sigma_x+\sigma_y}{2}\right)^2 + \left(\frac{\sigma_y-\sigma_x}{2}\right)^2\cos^2 2\theta + \cancel{2}\times\left(\frac{\sigma_x+\sigma_y}{\cancel{2}}\right)\times\left(\frac{\sigma_y-\sigma_x}{2}\right)\cos 2\theta + \left(\frac{\sigma_y-\sigma_x}{2}\right)^2\sin^2 2\theta}$$

$$= \sqrt{\left(\frac{\sigma_x+\sigma_y}{2}\right)^2 + \left(\frac{\sigma_y-\sigma_x}{2}\right)^2 + \frac{1}{2}(\sigma_x+\sigma_y)(\sigma_y-\sigma_x)\cos 2\theta}$$

$$= \sqrt{\frac{\sigma_x^2+\sigma_y^2+2\cancel{\sigma_x\sigma_y}+\sigma_y^2+\sigma_x^2-2\cancel{\sigma_y\sigma_x}}{4} + \frac{1}{2}(\sigma_y^2-\sigma_x^2)\cos 2\theta}$$

$$= \sqrt{\left(\frac{\sigma_x^2+\sigma_y^2}{2}\right) + \left(\frac{\sigma_y^2-\sigma_x^2}{2}\right)\cos 2\theta}$$

$$= \sqrt{\frac{\sigma_x^2+\sigma_y^2}{2} + \frac{\sigma_y^2-\sigma_x^2}{2}(1-2\sin^2\theta)}$$

$$= \sqrt{\cancel{\frac{\sigma_x^2}{2}}+\frac{\sigma_y^2}{2} + \frac{\sigma_y^2}{2} - \cancel{\frac{\sigma_x^2}{2}} - (\sigma_y^2-\sigma_x^2)\sin^2\theta}$$

$$= \sqrt{\sigma_y^2 - \sigma_y^2\sin^2\theta + \sigma_x^2\sin^2\theta}$$

$$= \sqrt{\sigma_x^2\sin^2\theta + \sigma_y^2(1-\sin^2\theta)}$$

$$\boxed{\sigma_R = \sqrt{\sigma_x^2\sin^2\theta + \sigma_y^2\cos^2\theta}}$$

If ϕ = Angle between σ_n & σ_r

$$\tan\phi = \frac{\tau}{\sigma_n}$$

$$= \frac{(\sigma_y - \sigma_x)\sin\theta\cdot\cos\theta}{\sigma_x \sin^2\theta + \sigma_y \cos^2\theta}$$

$$\tan\phi = \frac{\sigma_y - \sigma_x}{\sigma_x \tan\theta + \sigma_y \cancel{\cot}\theta}$$

(Dividing $\sin\theta.\cos\theta$ on both numerator & denomenator)

$$\boxed{\phi = \tan^{-1}\left[\frac{\sigma_y - \sigma_x}{\sigma_x \tan\theta + \sigma_y \cot\theta}\right]}$$

In order to determine the greatest obliquity or inclination to the normal of the resultant stress.

$$\frac{d}{d\theta}(\tan\phi) = 0$$

$$\sigma_x \sec^2\theta - \sigma_y \operatorname{cosec}^2\theta = 0$$

$$\tan\theta = \sqrt{\frac{\sigma_y}{\sigma_x}}$$

$$\tan\phi_{max} = \frac{\sigma_y - \sigma_x}{2\sqrt{\sigma_x \sigma_y}}$$

Unlike stresses

let σ_x = Tensile , σ_y = compressive

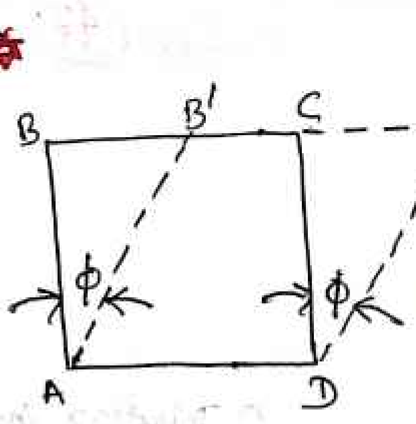
$$\sigma_n = \sigma_x \sin^2\theta - \sigma_y \cos^2\theta$$

~~see~~ $\Rightarrow \sigma_n = \left(\frac{\sigma_x - \sigma_y}{2}\right) \mp \left(\frac{\sigma_x + \sigma_y}{2}\right)\cos 2\theta$

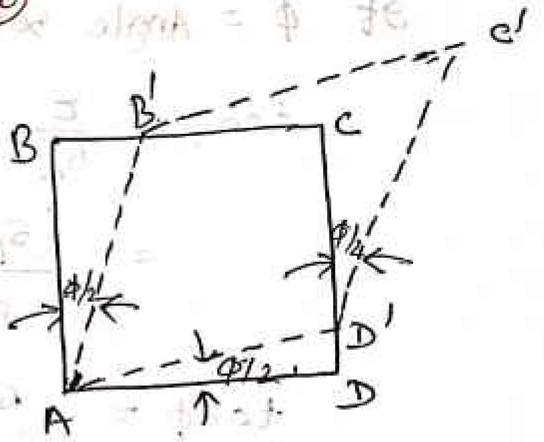
$$\tau = -\frac{1}{2}(\sigma_x + \sigma_y)\sin 2\theta$$

$$\tau_{max} = -\frac{1}{2}(\sigma_x + \sigma_y)$$

Note

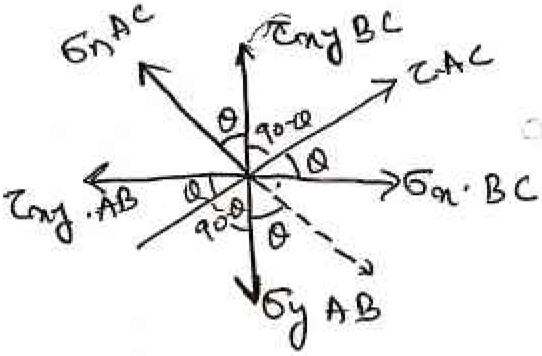


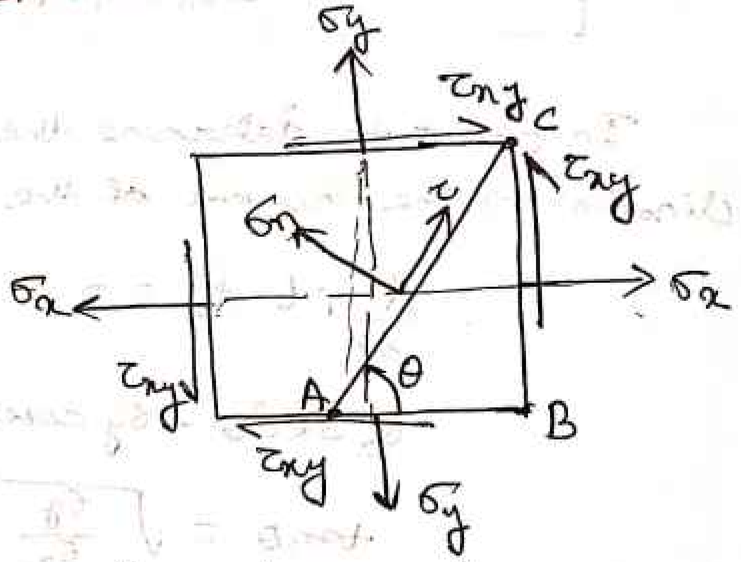
(Simple shear)



(Pure shear)

Biaxial stresses combined with shear stresses →





(Two dimensional complex stress system)

Resolving the forces along σ_n.

$$\sigma_n \cdot AC + \tau_{xy} BC \cos\theta - \sigma_x BC \cdot \sin\theta - \sigma_y AB \cos\theta + \tau_{xy} AB \sin\theta = 0$$

Dividing AC throughout

$$\sigma_n + \tau_{xy} \cdot \frac{BC}{AC} \cdot \cos\theta - \sigma_x \frac{BC}{AC} \cdot \sin\theta - \sigma_y \frac{AB}{AC} \cdot \cos\theta + \tau_{xy} \frac{AB}{AC} \cdot \sin\theta = 0$$

$$\sigma_n + \tau_{xy} \cdot \sin\theta \cdot \cos\theta - \sigma_x \cdot \sin\theta \cdot \sin\theta - \sigma_y \cdot \cos\theta \cdot \cos\theta + \tau_{xy} \cdot \cos\theta \cdot \sin\theta = 0$$

$$\sigma_n + 2\tau_{xy}\sin\theta\cdot\cos\theta - \sigma_x\sin^2\theta - \sigma_y\cos^2\theta = 0$$

$$\sigma_n = \sigma_x\sin^2\theta + \sigma_y\cos^2\theta - 2\tau_{xy}\sin\theta\cdot\cos\theta$$

$$= \sigma_x\left(\frac{1-\cos 2\theta}{2}\right) + \sigma_y\left(\frac{1+\cos 2\theta}{2}\right) - \tau_{xy}\sin 2\theta$$

$$\boxed{\sigma_n = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_y-\sigma_x}{2}\right)\cos 2\theta - \tau_{xy}\sin 2\theta}$$

Resolving along 'τ':

$$\tau\cdot AC + \sigma_x BC\cdot\cos\theta + \tau_{xy} BC\cdot\sin\theta = \tau_{xy} AB\cos\theta + \sigma_y AB\sin\theta$$

$$\tau = \tau_{xy}\frac{AB}{AC}\cdot\cos\theta - \tau_{xy}\frac{BC}{AC}\cdot\sin\theta + \sigma_y\frac{AB}{AC}\sin\theta - \sigma_x\frac{BC}{AC}\cdot\cos\theta$$

$$\tau = \tau_{xy}\cos^2\theta - \tau_{xy}\sin^2\theta + \frac{\sigma_y}{2}\cdot\sin 2\theta - \frac{\sigma_x}{2}\sin 2\theta$$

$$\boxed{\tau = \left(\frac{\sigma_y-\sigma_x}{2}\right)\sin 2\theta - \tau_{xy}\cos 2\theta}$$

σ_n to have max^m or min^m value :

$$\frac{d\sigma_n}{d\theta} = 0$$

Mohr's circle for Bi-axial stresses >

i) Like stresses:

For drawing the Mohr's circle, we have to determine the angle between the inclined plane & the σ_n stre

$$\sigma_n = \frac{(\sigma_x+\sigma_y)}{2} + \left(\frac{\sigma_y-\sigma_x}{2}\right)\cos 2\theta \quad \text{——— (1)}$$

$$\tau = \left(\frac{\sigma_y-\sigma_x}{2}\right)\sin 2\theta \quad \text{——— (2)}$$

$$\sigma_n - \left(\frac{\sigma_x+\sigma_y}{2}\right) = \left(\frac{\sigma_y-\sigma_x}{2}\right)\cos 2\theta \quad \text{——— (3)}$$

Squaring both sides

$$\left\{\sigma_n - \left(\frac{\sigma_x+\sigma_y}{2}\right)\right\}^2 = \left\{\left(\frac{\sigma_y-\sigma_x}{2}\right)\cos 2\theta\right\}^2 \quad \text{——— (4)}$$

sq. eqn (2) on both sides $\quad \tau^2 = \left(\frac{\sigma_y-\sigma_x}{2}\right)^2 \sin^2 2\theta$ ——— (5)

Adding (4) & (5)

$$\left\{\sigma_n - \left(\frac{\sigma_x+\sigma_y}{2}\right)\right\}^2 + \tau^2 = \left\{\left(\frac{\sigma_y-\sigma_x}{2}\right)\cos^2 2\theta\right\} + \left(\frac{\sigma_y-\sigma_x}{2}\right)^2 \sin^2 2\theta$$

$$= \left(\frac{\sigma_y-\sigma_x}{2}\right)^2 \left\{\cos^2 2\theta + \sin^2 2\theta\right\}$$

$$\left\{\sigma_n - \left(\frac{\sigma_x+\sigma_y}{2}\right)\right\}^2 + \tau^2 = \left(\frac{\sigma_y-\sigma_x}{2}\right)^2 \quad \text{——— (6)}$$

This represents the equation of circle having centre is at $\left(\frac{\sigma_x+\sigma_y}{2}, 0\right)$. The radius is $\frac{\sigma_y-\sigma_x}{2}$

$(x^2+y^2=r^2)$ $\{(x-a)^2+(y-b)^2=r^2$

Maximum principal stress,

$$\sigma_{p_1} = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Minimum principal stress,

$$\sigma_{p_2} = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Maximum shearing stress,

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Bending Moment. & Shear force

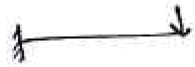
Shear force :- (S.F.)

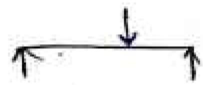
The shear force at the cross-section of a beam may be defined as the sum of the unbalanced vertical forces to the right or left of the section.

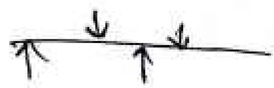
Bending moment :- (BM)

The bending moment at the cross section of a beam may be defined as the algebric sum of the moments of forces, to the right or left of the section.

Types of beam →

i) Cantilever beam 

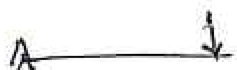
ii) Simply supported beam 

iii) overhanging beam 

iv) Rigidly fixed or built-in-beam 

v) Continuous beam.

Types of Loading →

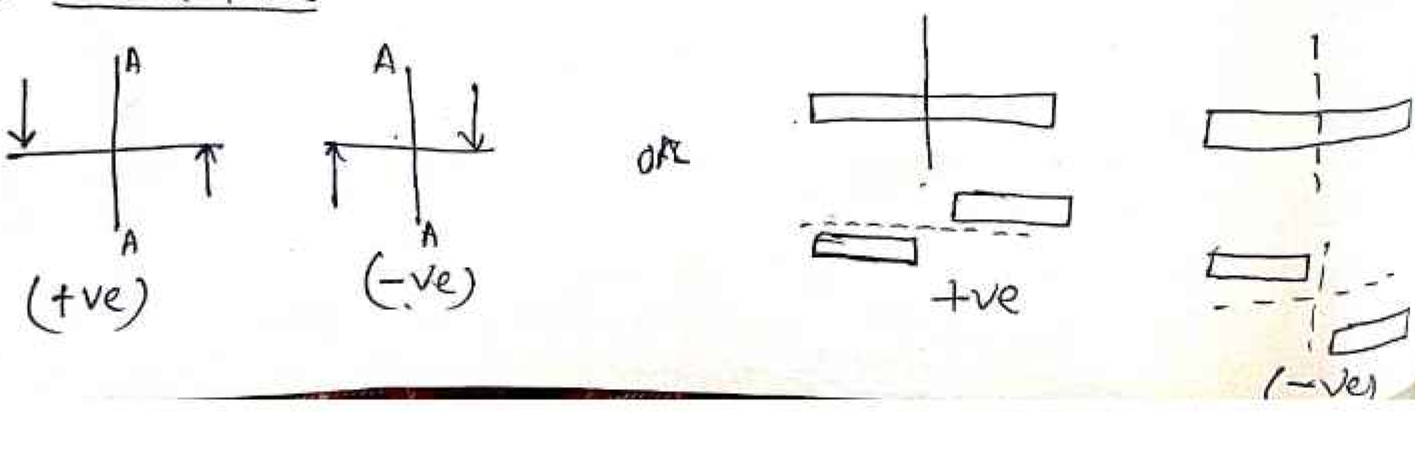
i) Concentrated or point load 

ii) Uniformly distributed Load 

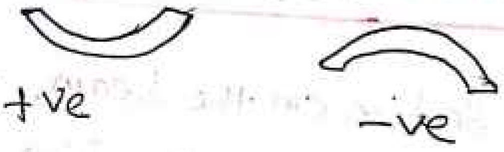
iii) Uniformly varying Load 

Sign convention

1) Shear force



2) Bending Moment



+ve BM = Sagging moment

-ve BM = hogging moment

Shear force (Sign convention)

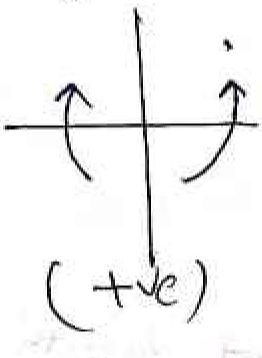
Shear force is the unbalanced vertical force, therefore it tends to slide one portion of the beam, upward or downward with respect to other.

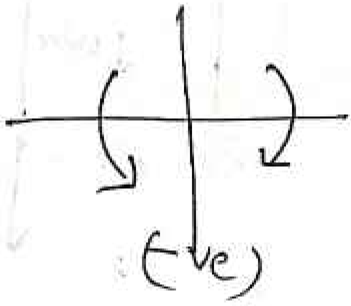
The shear force is said to be positive at a section, when the left hand portion tends to slide downward or the right hand portion tends to slide upward or in otherwards all the downward forces to the left of the section cause positive shear and those acting upwards cause negative shear.

Similarly, the shear force, is said to be negative at a section when the left hand portion tends to slide upwards or right hand portion tends to slide downward. or in otherwards, all the upward forces to the left of the section cause negative shear and those acting downward cause positive shear.

Bending Moment:

At sections, where the bending moment is such that it tends to bend the beam at that point to a curvature having concavity at the top is taken as positive. On the otherhand where the bending moment is such that it tends to bend the beam at that point to a curvature having convexity at the top is taken as negative.





Relation between Loading, Shear force & Bending moment

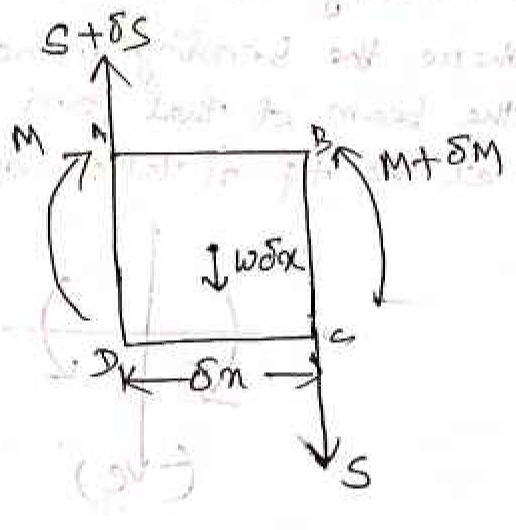
1) If there is a point Load at a Section on the beam, then the Shear force Suddenly changes. But the Bending moment remains the same. (i.e. S.F line is vertical)

2) If there is no load between two points, then the Shear force does not change. But the bending moment changes Linearly (B.M. line is an inclined straight line)

3) If there is uniformly distributed Load between two Points, then the shear force changes linearly (i.e. S.F. line is inclined straight line). But the bending moment changes according to the Parabolic Law (i.e. B.M. line is parabola)

4) If there is uniformly varying Load between two Points then shear force changes according to Parabolic Law (i.e. S.F. line will be parabola) But the bending moment changes according to cubic law.

General Relation between the Load, Shearing force & Bending Moment



let's consider a small portion of beam of length δx.

$\Sigma f_y = 0$

$S + \delta S = S + w\delta x$

$\delta S = w \delta x$

$w = \dfrac{\delta s}{\delta x}$

or $\boxed{w = \dfrac{ds}{dx}}$ ——— (1)

$\cancel{M + \delta M - M - S\frac{\delta x}{2} + (S + \delta S)\frac{\delta x}{2} = 0}$

$\cancel{\delta M - S\frac{\delta x}{2} + S\frac{\delta x}{2} + \delta S\frac{\delta x}{2}}$

Taking moment about Point 'A'.

$\cancel{M} + \delta M - \cancel{M} - S \cdot \delta x - w\,\delta x \cdot \frac{\delta x}{2} = 0$

$\delta M - S \cdot \delta x - \cancel{w\frac{(\delta x)^2}{2}} \to 0 = 0 \quad (\because \delta x \to 0)$

$\delta M = S\,\delta x$

$S = \dfrac{\delta M}{\delta x}$

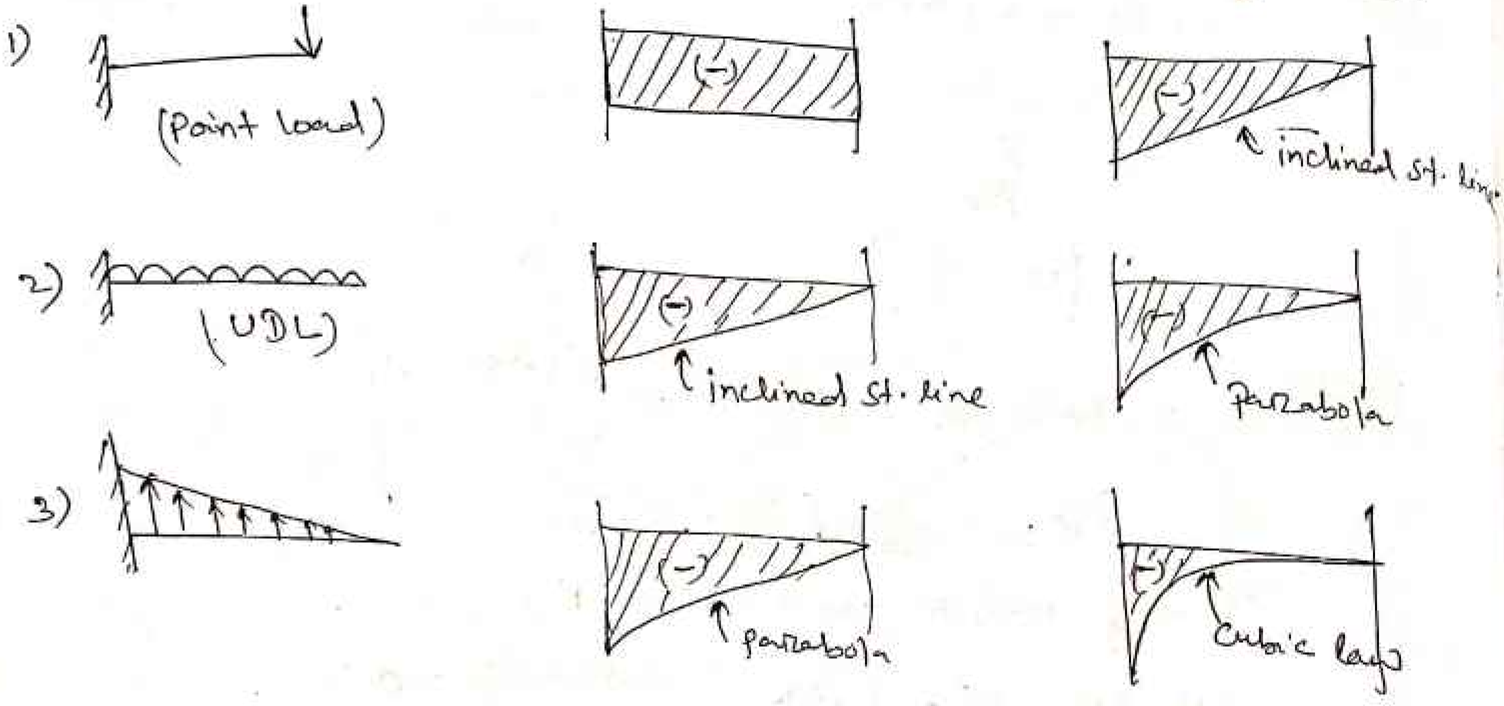
or $\boxed{S = \dfrac{dM}{dx}}$ ——— (2)

From equation (1) it is concluded that the rate of change of shear force at any section represents the rate of loading at the section.

From equation (2), it is concluded that the rate of change of Bending moment at any section represents shear force at that section.

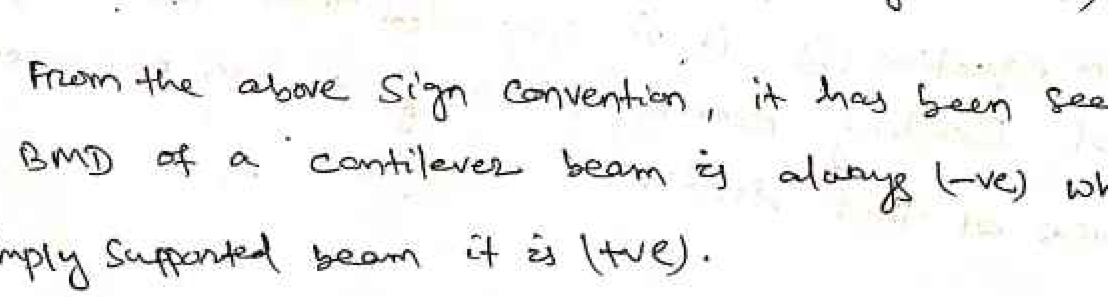
Nature of the line of SF & BM corresponds to Loads.

Types of load | Shear force | Bending Moment



Note: 1) During the calculation of end reactions of a simply supported beam, follow the method of moments. Take anticlockwise moment (+ve) & clockwise moment (−ve).

2)



From the above Sign convention, it has been seen that the BMD of a cantilever beam is always (−ve) whereas for a simply supported beam it is (+ve).

3) Bending moment will be maximum or minimum when $\frac{dM}{dx} = 0$ i.e $S = 0$, Thus at the section where S.F. is zero or changes sign, the B.M is either maximum or minimum.

Point of contraflexure →

An overhanging beam is analysed as combination of simply supported beam and cantilever beam. We have seen that the B.M. in a cantilever beam is -ve & the B.M. in a simply supported beam is +ve. So, in case of overhanging beam, a part is considered as cantilever & other part is simply supported beam i.e. there is a point in bending moment diagram where it changes ~~ch~~ sign. i.e +ve to -ve or vice versa. Such a point where B.M. changes its sign is known as Point of contraflexure.

Bending stresses in Beams →

The Bending moment at a section tends to bend or deflect the beam, and the internal stresses resist its bending. The process of bending stops, when every cross-section set up full resistance to the bending moment. The resistance offered by the internal stresses to the bending is called bending stress And the relevant theory is called theory of simple bending.

Assumptions in the theory of simple Bending →

1. The material of the beam is perfectly homogeneous & isotropic.

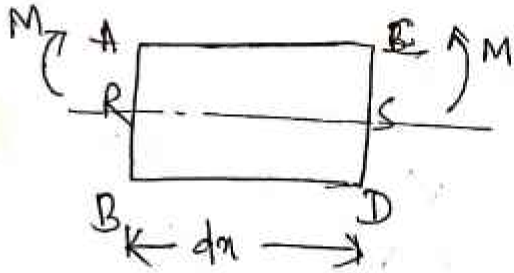
   Homogeneous : of the same kind throughout

   Isotropic : of equal elastic properties in all directions.

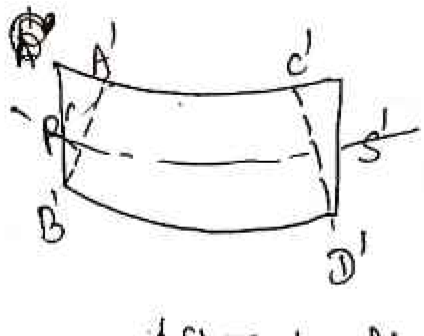
2. The beam material is stressed within its elastic limit i.e it obeys Hooke's Law.
3. The transverse sections, which were plane before bending, remain plane after bending also.
4. Each layer of the beam is free to expand or contract, independently of the layer, above or below it.
5. The value of 'E' is the same in tension & compression.

Theory of Simple Bending →

Let's consider a beam is subjected to BM 'M'. & the fibres are shown in fig. before & after bending.



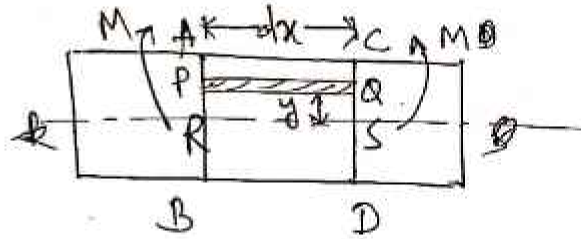
(Before Bending)



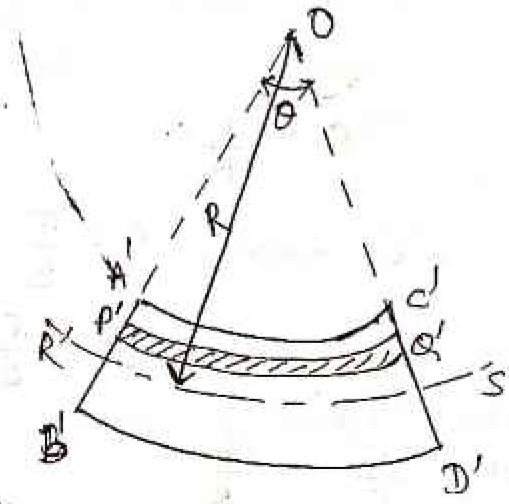
(After bending)

By this application of Bending Moment 'M' the upper fibre AC is under compression (A'C') & BD, the lower fibre is in tension (B'D'). But the fibre RS = R'S' i.e it is same before as well as after bending. The amount of compression & tension of the layer depends upon the position of RS. The layer RS, which is neither compressed nor stretched is known as neutral plane or neutral layer. This theory of bending is called theory of simple bending.

Bending Stress →



(RS = PQ)



Let's consider a small length 'dx' of a beam subjected to a bending moment. Let this small length of beam bend into an arc of a circle with 'O' as centre

M = Moment acting at the beam

θ = Angle subtended at centre by the arc.

R = Radius of curvature of the beam

y = Distance from RS to PQ.

$$\delta l = PQ - P'Q'$$

$$\varepsilon = \frac{\delta l}{l} = \frac{PQ - P'Q'}{PQ}$$

from the geometry OP'Q' & OR'S'

$$\frac{P'Q'}{R'S'} = \frac{R-y}{R}$$

$$\frac{P'Q'}{R'S'} = 1 - \frac{y}{R}$$

$$1 - \frac{P'Q'}{R'S'} = \frac{y}{R}$$

$$\frac{R'S' - P'Q'}{R'S'} = \frac{y}{R}$$

$$\frac{PQ - P'Q'}{PQ} = \frac{y}{R} \quad \left(\because \begin{array}{l} PQ = RS \\ RS = R'S' \\ \Rightarrow PQ = R'S' \end{array} \right)$$

$$\varepsilon = \frac{y}{R}$$

we know $E = \frac{\sigma}{\varepsilon}$

$$\varepsilon = \frac{\sigma}{E}$$

So, $\frac{\sigma}{E} = \frac{y}{R}$

$$\boxed{\frac{\sigma}{y} = \frac{E}{R}}$$

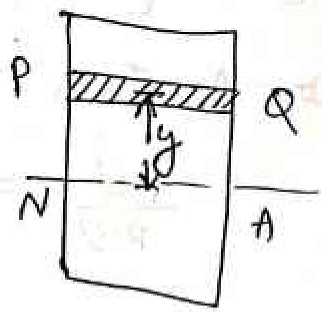
$$\Rightarrow \boxed{\sigma = \frac{E}{R} y}$$

Position of Neutral axis →

The layer or fibre of beam where there is no stress is acting called Neutral axis (RS).

On one side of Neutral axis there are compressive stress whereas on the other side there are tensile stress

Let δa = Area of layer PQ.



Intensity of stress in layer PQ,

$$\sigma = y \times \frac{E}{R}$$

Total stress on Layer PQ OR normal force on PQ

$$= \text{Intensity of stress} \times \text{Area}$$

$$= y \times \frac{E}{R} \times \delta a$$

Total stress of section

$$= \sum y \frac{E}{R} \delta a$$

$$= \frac{E}{R} \sum y \delta a$$

Since the section is in equilibrium, therefore total stress, from top to bottom; must equal to zero.

$$\frac{E}{R} \sum y \delta a = 0$$

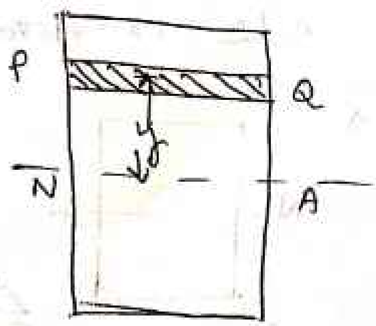
$$\boxed{\sum y \delta a = 0}$$

From the above, it is thus obvious, that the neutral axis of the section will be so located that moment of the entire area about the axis is zero.

Imp

Moment of Resistance →

On one side of the neutral axis there are compressive stresses and on other side there are tensile stresses. These stresses form a couple whose moment must be equal to the external moment (M). The moment of this couple, which resists the external ~~loading in~~ bending moment is known as Moment of Resistance.



let δa = Area of the Layer PQ.

Intensity of stress in the layer PQ.

$$\sigma = y \times \frac{E}{R}$$

Total stress in the layer PQ OR normal force on PQ.

$$= y \times \frac{E}{R} \times \sigma a$$

Moment of this total force on PQ about neutral axis = $F \times y$ (64)

$$= y \times \frac{E}{R} \times \delta a \times y$$

$$= \frac{E}{R} y^2 \delta a$$

The algebraic sum of all such moments about the neutral axis must be equal to M. Therefore

$$M = \Sigma \frac{E}{R} y^2 \delta a$$

$$= \frac{E}{R} \Sigma y^2 \delta a.$$

The expression $\Sigma y^2 \delta a$ = moment of inertia of the area of the whole section about neutral axis.

Therefore, $M = \frac{E}{R} I$

$$\boxed{\frac{M}{I} = \frac{E}{R}} \quad \text{——— (a)}$$

We have derived $\frac{\sigma}{y} = \frac{E}{R}$ ——— (b)

From (a) & (b) we can write

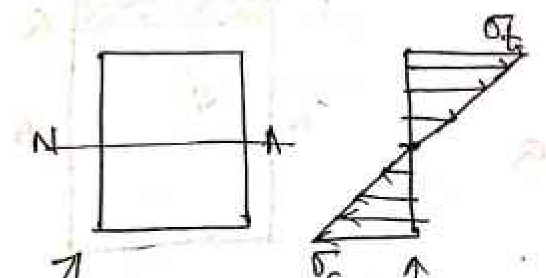
$$\boxed{\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}}$$

pure bending equation.

Distribution of Bending stress →

→ on neutral axis $\sigma = 0$, on one side it is tensile & another side compressive.

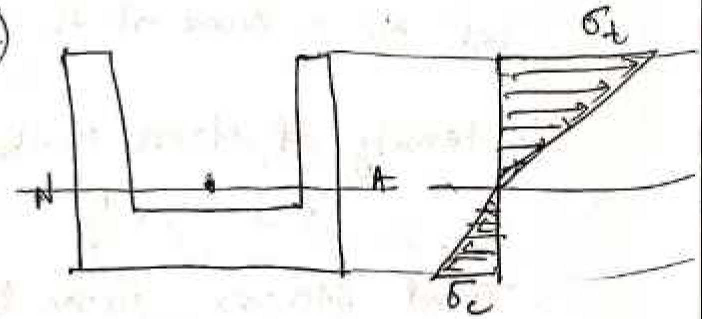
1)



Cross section of beam

Bending stress distribution

2)



Imp

Section Modulus →

from the relation $\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$M = \sigma \times \frac{I}{y}$$

We know, stress is directly proportional to the distance from C.G. to the extreme fibre of the section.

$$M = \sigma_{max} \times \frac{I}{y_{max}}$$

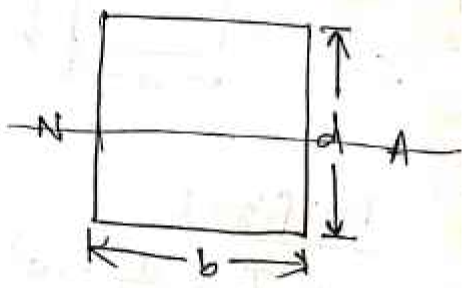
The term $\frac{I}{y_{max}}$ is known as Section Modulus & it is denoted as Z.

$$\boxed{Z = \frac{I}{y_{max}}}$$

$$M = \sigma \times Z$$

The strength of the beam section depends mainly on the Section Modulus.

1) Section Modulus for Rectangular Section →

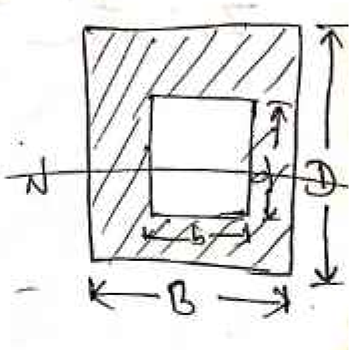


$$I_{CG} = \frac{1}{12} \times bd^3$$

$$y_{max} = \frac{d}{2}$$

$$Z = \frac{I}{y_{max}} = \frac{\frac{1}{12} bd^3}{\frac{d}{2}} = \frac{bd^2}{6}$$

2) Section modulus for hollow Rectangular section →

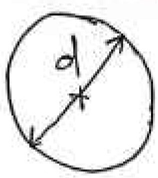


$$I_{CG} = \frac{1}{12} BD^3 - \frac{1}{12} bd^3$$

$$y_{max} = D/2$$

$$Z = \frac{I}{y_{max}} = \frac{\frac{1}{12}(BD^3 - bd^3)}{\frac{D}{2}} = \frac{BD^3 - bd^3}{6D}$$

3) Section modulus for circular section → (7)

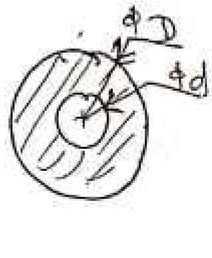


$$I = \frac{\pi}{64} d^4$$

$$y_{max} = \frac{d}{2}$$

$$Z = \frac{I}{y_{max}} = \frac{\frac{\pi}{64} d^4}{d/2} = \frac{\pi}{32} d^3$$

4) Section modulus for hollow circular section →



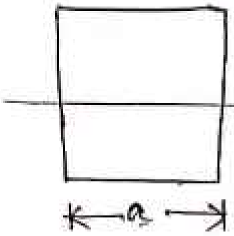
$$I = \frac{\pi}{64} D^4 - \frac{\pi}{64} d^4$$

$$y_{max} = D/2$$

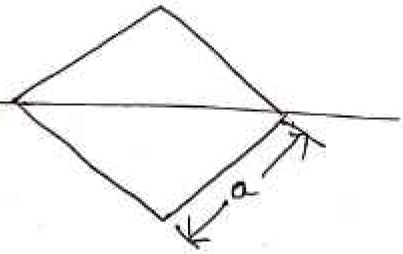
$$Z = \frac{I}{y_{max}} = \frac{\frac{\pi}{64}(D^4 - d^4)}{D/2} = \frac{\pi}{32}\left(\frac{D^4 - d^4}{D}\right)$$

Q For a given stress, compare the moments of resistance of a beam of a square section, when placed i) with two sides horizontal ii) with its diagonal horizontal.

Solⁿ:–



(fig-1)



(fig-2)

For fig-1

$$I = \frac{1}{12} a^4$$

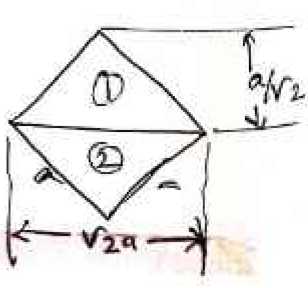
$$y_{max} = \frac{a}{2}$$

$$Z_1 = \frac{I}{y_{max}} = \frac{a^4}{12} \Big/ \frac{a}{2} = \frac{a^3}{6}$$

fig-2

$$I = I_1 + I_2$$

$$= 2I_1 \quad (I_1 = I_2)$$

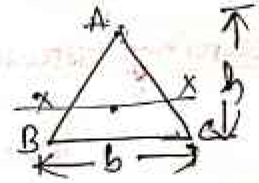


(8)

$$I_1 = \frac{1}{12} \times \sqrt{2}a \left(\frac{a}{\sqrt{2}}\right)^3$$

$$= \frac{1}{12} \times \cancel{\sqrt{2}}a \times \frac{a^3}{2\cancel{\sqrt{2}}}$$

$$= \frac{a^4}{24}$$



$I_{BC} = \frac{1}{12} bh^3$

$I_{xx} = \frac{1}{36} bh^3$

$I_A = \frac{1}{4} bh^3$

$$I = 2 I_1$$

$$= 2 \times \frac{a^4}{24} = \frac{a^4}{12}$$

$$y_{max} = \frac{a}{\sqrt{2}}$$

$$Z_2 = \frac{I}{y_{max}} = \frac{a^4}{12} \Big/ \frac{a}{\sqrt{2}} = \frac{a^4}{12} \times \frac{\sqrt{2}}{a}$$

$$= \frac{a^3}{6\sqrt{2}}$$

$$M_1 = \sigma Z_1$$

$$M_2 = \sigma Z_2$$

$$\frac{M_1}{M_2} = \frac{\cancel{\sigma} Z_1}{\cancel{\sigma} Z_2} = \frac{Z_1}{Z_2} = \frac{a^3}{6} \Big/ \frac{a^3}{6\sqrt{2}} = \sqrt{2}$$

$$\boxed{\frac{M_1}{M_2} = \sqrt{2} = 1.414}$$

Strength of a section → (Flexural strength)

It is also termed as strength of a section, which means the moment of resistance offered by it.

$$\boxed{M = \sigma Z}$$

Torsion of Circular Shafts (1)

Whenever a shaft is subjected to a turning force during the transmit of power, due to this turning force, a torque may developed (turning force x distance between the point of application of the force & axis of the shaft). This torque is also known as turning moment or twisting moment & the shaft is said to be subjected to torsion. Due to this torque, every cross-section of the shaft is subjected to some shear stress.

Assumption for shear stress in a circular shaft subjected to Torsion →

1) The material of the shaft is uniform throughout.
2) The twist along the shaft is uniform.
3) Normal cross section of the shaft which were plane & circular before twist, remain plane & circular even after twist.
4) All diameters of the normal cross-section which were straight before twist, remain ~~plane & circular eve~~ straight with their magnitude unchanged after twist.

Torsional stresses and strains →

Consider a circular shaft fixed at one end and subjected to a torque at the other end.

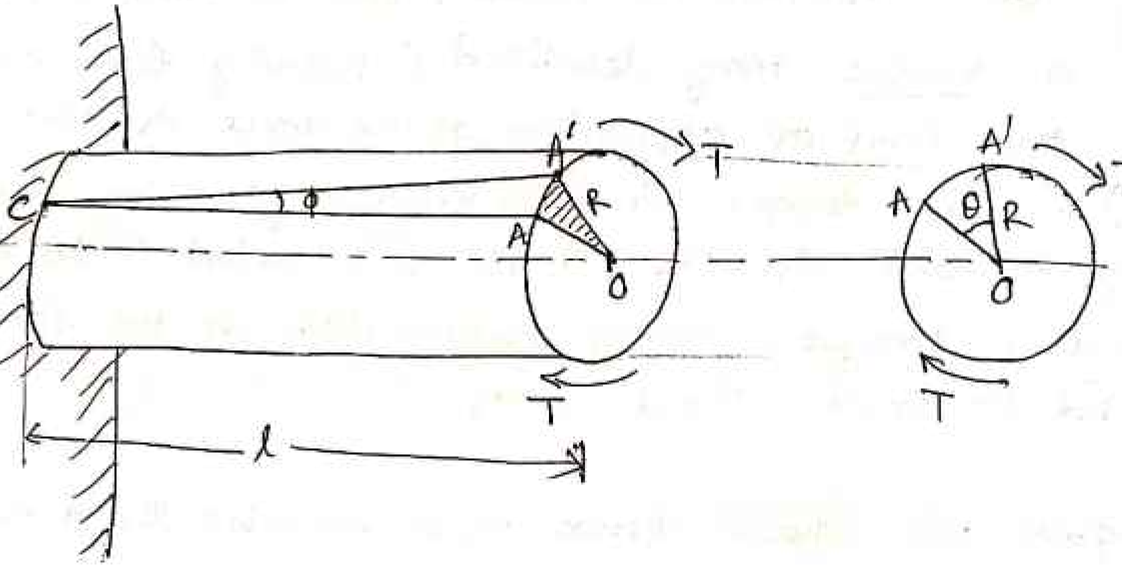
Let T = Torque in N-mm

l = Length of shaft in mm.

R = Radius of circular shaft

As a result of this torque, every cross-section of the shaft will be subjected to shear stress.

Let the line CA on the surface of the shaft be deformed to CA' and OA to OA' as shown in fig.



Let $\angle ACA' = \phi$ in degrees

$\angle AOA' = \theta$ in radians

τ = Shear stress induced at the surface

C = Modulus of rigidity or torsional rigidity of the shaft material

Shear strain $= \frac{AA'}{l} = \tan\phi$

since ϕ is very small, so $\tan\phi = \phi$

Also, the arc $AA' = R\theta$

$$\phi = \frac{AA'}{l} = \frac{R\theta}{l} \quad \text{——— (1)}$$

If τ is the intensity of shear stress on the outermost Layer & C the modulus of rigidity of the shaft, then

$$\phi = \frac{\tau}{C} \quad \text{——— (2)} \quad \left(\because \; C \text{ or } G = \frac{\tau}{\phi} = \frac{\text{Shear stress}}{\text{Shear strain}} \right)$$

Equating eqⁿ (1) & (2)

$$\frac{\tau}{C} = \frac{R\theta}{l} \Rightarrow \boxed{\frac{\tau}{R} = \frac{C\theta}{l}}$$

If τ_x = Intensity of shear stress on any layer at a distance x from centre of shaft

then, $\boxed{\frac{\tau_x}{x} = \frac{\tau}{R} = \frac{C\theta}{l}}$

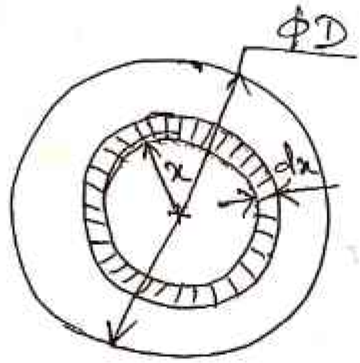
Strength of a Solid Shaft →

Strength of a ~~solid~~ shaft means the maximum torque or power, it can transmit.

Let R = Radius of shaft

τ = Shear stress developed in the outermost Layer of the shaft.

Consider a shaft subjected to a torque T. Let's consider an elementary area of 'da' of thickness dx at a distance x from the centre.



$$da = 2\pi x\, dx$$

Shear stress at this section

$$\tau_x = \tau \times \frac{x}{R} \qquad \left(\because \frac{\tau_x}{x} = \frac{\tau}{R}\right)$$

Turning force = Stress x Area

$$T_f = \tau_x \times da$$

$$= \tau \cdot \frac{x}{R} \cdot da$$

$$= \tau \cdot \frac{x}{R} \cdot 2\pi x\, dx$$

$$= \frac{2\pi\tau}{R} x^2 dx$$

Turning Moment of this element

dT = Turning force x distance of the element from centre

$$= T_f \times x$$

$$= \frac{2\pi\tau}{R} x^2 dx \times x$$

$$dT = \frac{2\pi\tau}{R} x^2 \, dx$$

Total Torque $\quad T = \int_0^R dT$

$$= \int_0^R \frac{2\pi\tau}{R} x^3 \, dx$$

$$= \frac{2\pi\tau}{R} \int_0^R x^3 \, dx$$

$$= \frac{2\pi\tau}{R} \left| \frac{x^4}{4} \right|_0^R$$

$$= \frac{2\pi\tau}{4R} (R^4 - 0)$$

$$= \frac{\pi\tau R^3}{2}$$

$$= \frac{\pi\tau \times \left(\frac{D}{2}\right)^3}{2} \qquad \left(\because R = \frac{D}{2}\right)$$

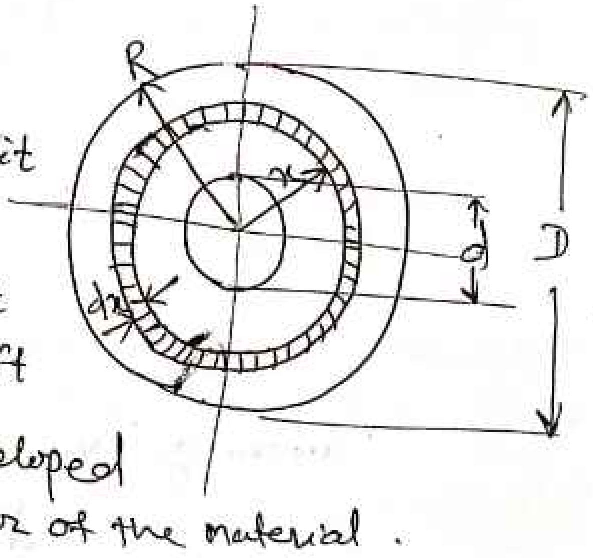
$$= \frac{\pi}{16} \tau D^3$$

Imp

$$\boxed{T = \frac{\pi}{16} \tau D^3}$$

Strength of a Hollow Shaft →

It means the maximum torque or power a hollow shaft can transmit from one pulley to another.



Let R = outer radius of the shaft

r = Inner radius of the shaft

τ = Maximum shear stress developed in the outer most layer of the material.

Now consider an elementary ring of thickness 'dx' at a distance 'x' from the centre

We know, $da = 2\pi x \cdot dx$

(where da = Area of the ring)

Shear stress at this Section $\tau_x = \tau \times \frac{x}{R}$

Turning force at this Section = Stress x area

$$T_f = \tau_x \times da$$
$$= \tau \times \frac{x}{R} \times 2\pi x \cdot dx$$
$$= \frac{2\pi\tau}{R} x^2 dx$$

Turning moment of this element

$dT = T_f \times$ distance of the element from the axis of shaft

$$= \frac{2\pi\tau}{R} x^2 dx \times x$$
$$= \frac{2\pi\tau}{R} x^3 dx$$

Total Torque, $T = \int_r^R dT$

$$= \int_r^R \frac{2\pi\tau}{R} x^3 dx$$
$$= \frac{2\pi\tau}{R} \int x^3 dx$$
$$= \frac{2\pi\tau}{R} \left| \frac{x^4}{4} \right|_r^R$$
$$= \frac{2\pi\tau}{4R} \left(R^4 - r^4 \right)$$
$$= \frac{\cancel{2}\pi\tau}{\cancel{2}4 \times \frac{D}{2}} \left\{ \left(\frac{D}{2}\right)^4 - \left(\frac{d}{2}\right)^4 \right\} \qquad \left(\because R = D/2,\ r = d/2\right)$$
$$= \frac{\pi}{16} \tau \left[\frac{D^4 - d^4}{D} \right]$$

$$\boxed{T = \frac{\pi}{16} \tau \left[\frac{D^4 - d^4}{D} \right]}$$

Power transmitted by a shaft →

The main purpose of shaft is to transmit power from the shaft to another. Now considering a rotating shaft, which transmits power from one of its ends to another.

let N = No. of revolutions per minute

T = Average torque in kN-m.

workdone per minute = force x Distance

$= T \times 2\pi N$ (1 revol = 2π, N rev = $2\pi N$)

$= 2\pi NT$

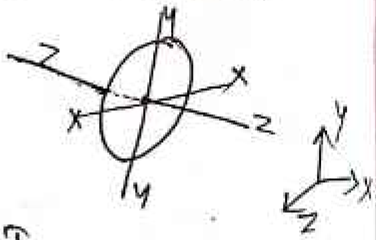
workdone per second $= \frac{2\pi NT}{60}$ kN-m

power transmitted = work done in kN-m/sec.

$$\boxed{P = \frac{2\pi NT}{60} \text{ kW}}$$

Polar Moment of Inertia →

The moment of inertia of a plane area, with respect to an axis perpendicular to the plane of fig. is called polar moment of inertia with respect to the point where the axis intersects the plane



I_{zz} = Polar moment

we know, $\frac{\tau}{R} = \frac{C\theta}{l} \Rightarrow \tau = \frac{C\theta}{l} \cdot R$ —— (1)

$T = \frac{\pi}{16} \tau D^3$

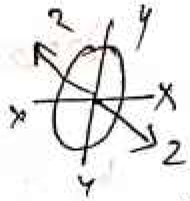
$\tau = \frac{16T}{\pi D^3}$ —— (2)

From (1) & (2)

$$\frac{16T}{\pi D^3} = \frac{C\theta}{l} \cdot R$$

$$\frac{T}{\frac{\pi}{16} D^3 R} = \frac{C\theta}{l}$$

$$\frac{T}{\frac{\pi}{16}D^3 \times \frac{D}{2}} = \frac{C\theta}{l}$$



$I_z = I_x + I_y$

& $I_x = I_y = \frac{\pi d^4}{64}$

$J = I_z = \frac{\pi d^4}{32}$

$$\frac{T}{\frac{\pi}{32}D^4} = \frac{C\theta}{l}$$

$$\boxed{\frac{T}{J} = \frac{C\theta}{l}} \text{ ——— } (3)$$

Where J = Polar moment of inertia $= \frac{\pi}{32}D^4$

we know $\frac{\tau}{R} = \frac{C\theta}{l}$ ——— (4)

from (3) & (4) we can write $\boxed{\frac{T}{J} = \frac{\tau}{R} = \frac{C\theta}{l}}$

The above eq^n is known as Torsion equation.

Note #1) $J_{solid} = \frac{\pi}{32}D^4$

$J_{hollow} = \frac{\pi}{32}(D^4 - d^4)$

Note-2) The term $\frac{J}{R}$ = Torsional section modulus or Polar modulus.

$Z_p = J/R$

Cor:

Z_p for solid shaft.

$$Z_p = \frac{J}{R}$$

$$Z_p = \frac{\frac{\pi}{32}D^4}{\frac{D}{2}} = \frac{\pi}{16}D^3$$

Z_p for hollow shaft

$$Z_p = \frac{J}{R} = \frac{\frac{\pi}{32}(D^4 - d^4)}{D/2}$$

$$Z_p = \frac{\pi}{16}\left(\frac{D^4 - d^4}{D}\right)$$

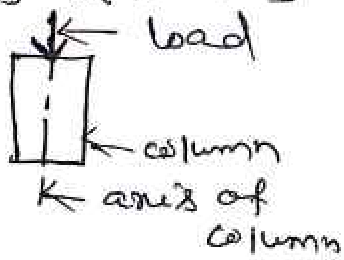
Combined and Bending stresses ①

Column :

A vertical member, subjected to an axial compressive force known as column.

Axial load on column :

When the compressive load will act along its axis, then that load is known as axial load on column.

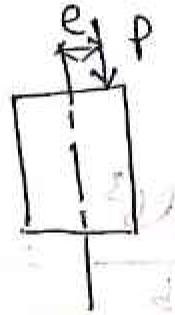


Eccentric Load on column :

When the load will act rather than axis of column, is known as eccentric load on column i.e. the line of action of load does not coincide with the axis of column, is known as eccentric load on column.

Note : 1) when eccentric load will act on column it is subjected to

→ Direct load
→ moment due to eccentric load.

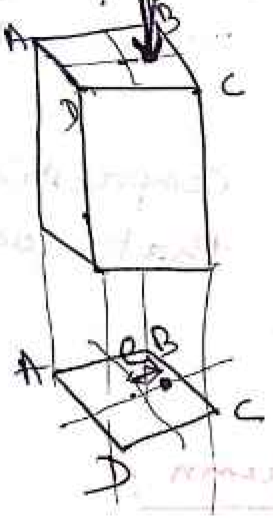


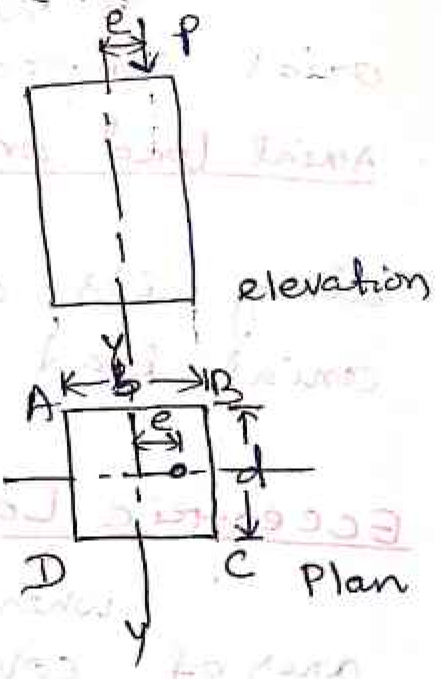
P = load
e = eccentricity

2) Eccentric load causes a direct stress as well as bending stress on column.

Symmetrical columns with eccentric loading about one axis

Let's consider a column ABCD is subjected to eccentric load about one axis (Y-Y axis) as shown in fig.





P = Load acting on column

e = eccentricity of load

b = width of column

d = Thickness of column

C.S. Area of column $A = b \times d$

moment of Inertia of column about an axis through its centre of gravity & parallel to axis about which the load is eccentric

$$I = \frac{1}{12} d b^3$$

$$y = b/2$$

Section modulus $Z = \frac{I}{y} = \frac{\frac{1}{12} d b^3}{\frac{b}{2}} = \frac{1}{6} d b^2$

Direct stress on column due to load,

$$\boxed{\sigma_d = \frac{P}{A}}$$

Moment due to load $M = P \cdot e$

From pure bending equation

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

$$\frac{M}{I} = \frac{\sigma_b}{y} \qquad (\sigma_b = \text{Bending stress})$$

$$\sigma_b = \frac{M}{I} \times y$$

$$= \frac{M}{(I/y)} = \frac{M}{Z}$$

$$\sigma_b = \frac{M}{I} \times y$$

at extreme fibre, $y = b/2$

$$\sigma_b = \frac{P \cdot e}{\frac{1}{12} d b^3} \times \frac{b}{2}$$

$$= \frac{6 P \cdot e}{d b^2} = \frac{6 Pe}{db \cdot b} = \frac{6 Pe}{A \cdot b}$$

$$\boxed{\sigma_b = \frac{6 Pe}{A b}}$$

Note: Eccentric load causes a direct stress as well as bending stress.

So, Total stress at extreme fibre,

$$\sigma = \sigma_d + \sigma_b$$

(4)

$$\sigma = \sigma_d \pm \sigma_b$$

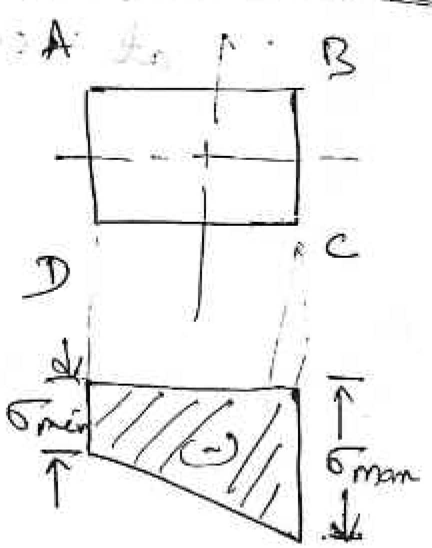
$$\boxed{\sigma = \frac{P}{A} \pm \frac{6Pe}{Ab}}$$ in terms of eccentricity

$$\boxed{\sigma = \frac{P}{A} \pm \frac{M}{Z}}$$ in terms of modulus of section

→ (+ve) or (−ve) sign will depend upon the position of fibre with respect to eccentric load.

→ From the fig. maximum stress will occur at corner B & C whereas minimum stress will occur at A & D. Because the load 'P' is nearer to corner A & D. Because the load 'P' is nearer to B & C (max^m stress) & far away from A & D (min^m stress)

Stress Distribution



(Throughout the section compressive stress will occur)

$$\boxed{\begin{aligned} \sigma_{max} &= \frac{P}{A} + \frac{6Pe}{Ab} \\ \text{or } \sigma_{max} &= \frac{P}{A} + \frac{M}{Z} \end{aligned}}$$

$$\boxed{\begin{aligned} \sigma_{min} &= \frac{P}{A} - \frac{6Pe}{Ab} \\ \text{or, } \sigma_{min} &= \frac{P}{A} - \frac{M}{Z} \end{aligned}}$$

Corollary

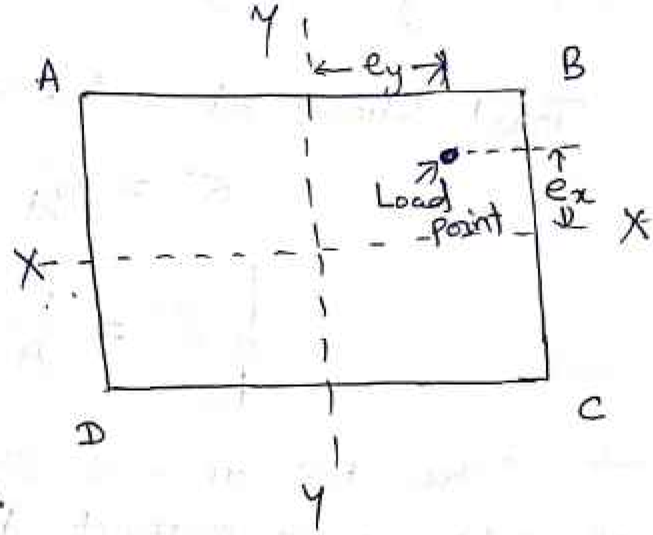
1) If $\sigma_d > \sigma_b$. (compressive stress throughout section)

2) If $\sigma_d = \sigma_b$ (maximum stress will be $2\sigma_d$ & minimum stress is zero)

3) If $\sigma_d < \sigma_b$ (stress will change its sign i.e partly compressive & partly tensile)

Symmetrical columns with Eccentric Loading about two axes :

Let's consider a column ABCD is subjected to a load with eccentricity about two axes (X & Y)



let P = Load on column

A = C.S area of column

e_x = Eccentricity of load about X-X axis

e_y = Eccentricity of load about Y-Y axis

M_x = moment about X-X axis

M_y = moment about Y-Y axis

I_{xx} = M.I. about X-X axis

I_{yy} = M.I. about Y-Y axis

The effect of such Load may be split up into three parts.

1) Direct Stress on column due to load

$$\sigma_d = \frac{P}{A}$$

2) Bending stress due to eccentricity e_x.

$$\sigma_{bx} = \frac{M_x}{I_{xx}} \cdot y$$

3) Bending stress due to eccentricity e_y.

$$\sigma_{by} = \frac{M_y}{I_{yy}} \cdot x$$

$M_{xx} = P \cdot e_x$

$M_y = P \cdot e_y$

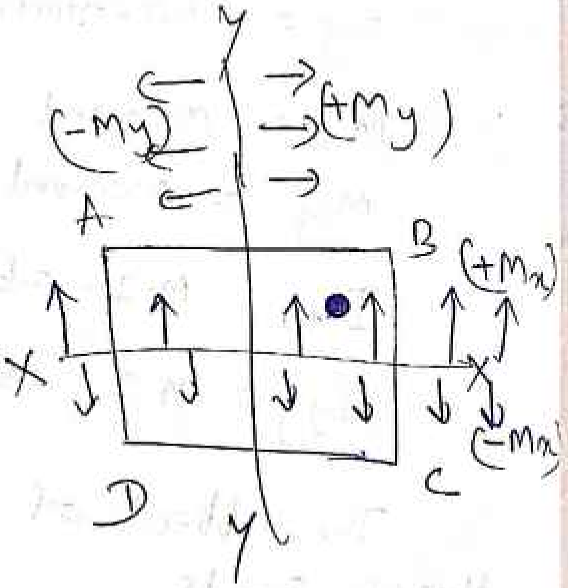
Total Stress at extreme fibre,

$$\sigma = \sigma_d \pm \sigma_{bx} \pm \sigma_{by}$$

$$\boxed{\sigma = \frac{P}{A} \pm \frac{M_x}{I_{xx}} \cdot y \pm \frac{M_y}{I_{yy}} \cdot x}$$

→ The +ve or −ve sign depends upon the Position of fibre with respect to Load.

→ The stress at pt. B will be maximum whereas minimum at pt. D.



Sign of M_x & M_y

Points	M_x	M_y
A	+ve	−ve
B	+ve	+ve
C	−ve	+ve
D	−ve	−ve

Crippling or Buckling Load →

The Load at which the column just buckles is called buckling Load, or critical load or crippling Load.

Euler's Column Theory →

$\frac{l_e}{k}$ = Slenderness ratio

where l_e = Length of column (equivalent length)
k = Radius of gyration

when $\frac{l_e}{k} > 80$, Long column

$\frac{l_e}{k} < 80$, Short column

Euler's Column Theory derives the crippling load of a long column under various end conditions..

$$\boxed{\text{Crippling Load } P = \frac{\pi^2 EI}{l_e^2}}$$

Sl no.	End Conditions	Equivalent length (l_e) l = actual length	Crippling Load P
1	Both end hinged	$l_e = l$	$P = \frac{\pi^2 EI}{(l_e)^2} = \frac{\pi^2 EI}{l^2}$
2	one end fixed & other end free	$L_e = 2l$	$P = \frac{\pi^2 EI}{(2l)^2} = \frac{\pi^2 EI}{4l^2}$
3	Both end fixed	$L_e = \frac{l}{2}$	$P = \frac{\pi^2 EI}{(l/2)^2} = \frac{4\pi^2 EI}{l^2}$
4	one end fixed and other hinged	$L_e = \frac{l}{\sqrt{2}}$	$P = \frac{\pi^2 EI}{(l/\sqrt{2})^2} = \frac{2\pi^2 EI}{l^2}$